(Some new results in) Higher Teichmüller Theory

(joint w/ D Alessandrini, N. Tholozan, A. Wienhard + G. Martone, F. Mazzoli, T. Zhang)

Sara Maloni (University of Virginia)

PLAN for TODAY:

① (G, X) - STRUCTURES & Teichmüller space

② QUASI-FUCHSIAN MANIFOLDS & PLEATED SURFACES

③ Higher Teichmuller spaces & Anosov representations

④ Few new results

1. (G,X)- STRUCTURES & Teichmüller space
2. QUASI-FUCHSIAN MANIFOLDS
3. Higher Teichmuller spaces & Anosov representations
4. Few new results

What is a GEOMETRY?

F. KLEIN

DEF Geometry $= (G, X)$ where

$X =$ conn. manifold (MODEL MFD)

$G =$ Lie gp acting TRANSITIVELY on X

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EXAMPLES

- $(\mathrm{Isom}(\mathbb{R}^n), \mathbb{R}^n)$ EUCLIDEAN geometry
- $(\mathrm{Aff}(\mathbb{R}^n), \mathbb{R}^n)$ AFFINE geometry
- $(\mathrm{Isom}(\mathbb{H}^n), \mathbb{H}^n)$ HYPERBOLIC geometry
- $(PSL_2\mathbb{C}, \mathbb{CP}^1)$ cx. proj. geometry
- $(PSL_3\mathbb{R}, \mathbb{RP}^2)$ REAL proj. geometry

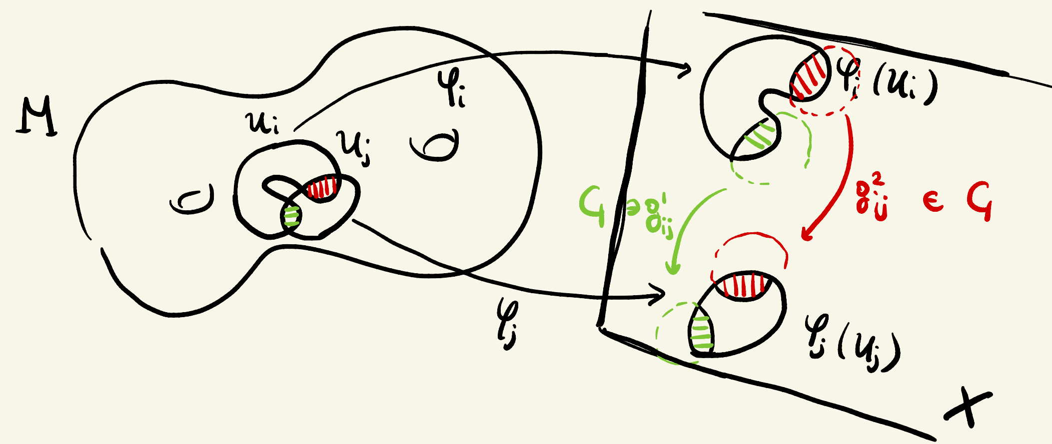
...

What is a GEOM. STRUCTURE on a MFD ?

DEF A (G,X)-ATLAS on mfd M is $((U_i),(\varphi_i))$

where • (U_i) = open covering of M "CHARTS"

• $\varphi_i : U_i \to X$ local diffeo st $\forall$ c.c. k of $U_i \cap U_j$ $\exists\, g^k \in G$ st $\varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(k)} = g^k\big|_{\varphi_i(k)}$.



What is a GEOM. STRUCTURE on a MFD ?

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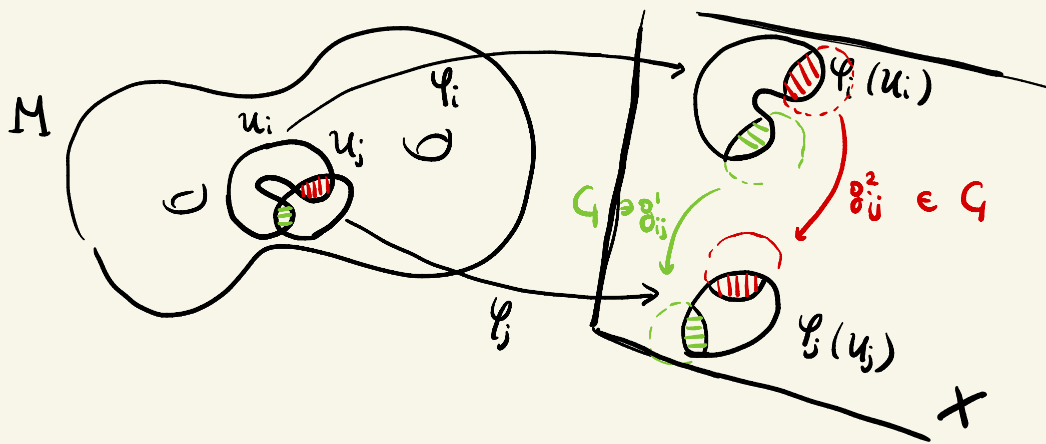
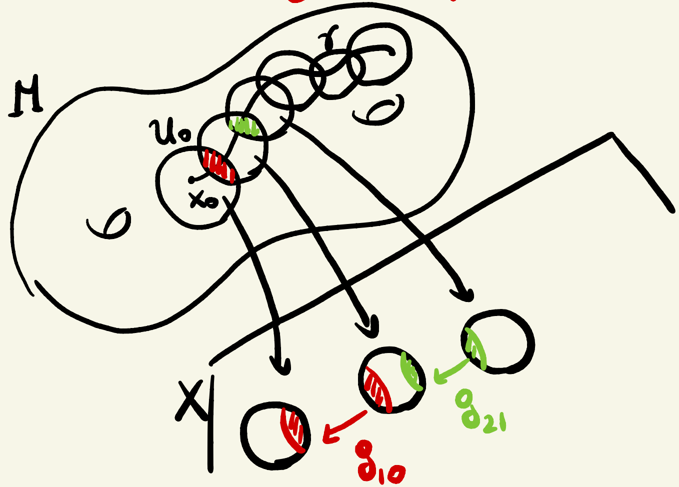


<u>DEF</u> A <u>(G,X)-STRUCTURE</u> on M is a MAXIMAL (G,X)-ATLAS on X.

Developing Map & Holonomy



Given a (G,X)-str on M, we can define (Dev, ρ):

- $\mathrm{Dev}: \tilde{M} \longrightarrow X$ local diffeo

  DEVELOPING MAP

- $\rho: \pi_1 M \longrightarrow G$ homom

  HOLONOMY MAP

st Dev is ρ-EQUIVARIANT (i.e. $\forall x \in \tilde{M}$, $\forall \gamma \in \pi_1 M$

$$\mathrm{Dev}(\gamma \cdot x) = \rho(\gamma) \cdot \mathrm{Dev}(x)\,)$$


EHRESMANN


THURSTON

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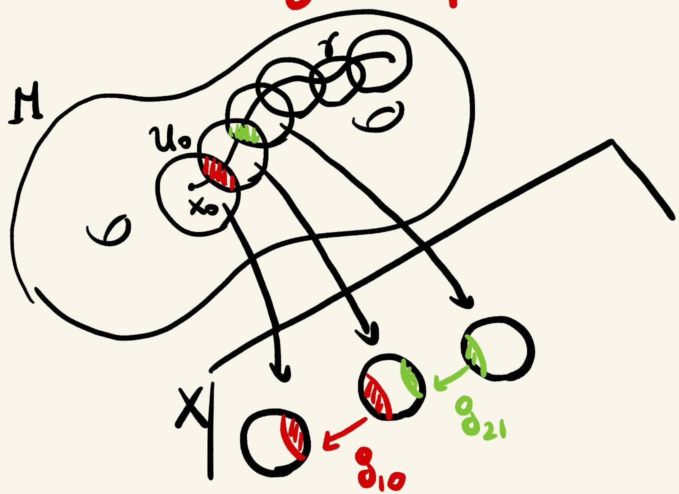

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- (Dev, ρ) is WD up to the action of G: $\forall A \in G$

$$A \cdot (\mathrm{Dev}, \rho) = (\mathrm{Dev} \circ A, A \rho(\cdot) A^{-1})$$

& such pair ! det the geom. str.

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QUESTIONS: ① Given M & given (G,X), can we endow M w/ a (G,X)-str ?

② If yes, in how many "different" ways we can do it?

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EXAMPLES ① $M = \mathbb{T}^2$, $(G,X) = (\mathrm{Isom}(\mathbb{R}^2), \mathbb{R}^2)$

$$\mathrm{Teich}_{\mathbb{R}^2}(\mathbb{T}^2) = \Big\{ (Y, \phi) \mid Y = \text{Euclidean str on } \mathbb{T}^2 \text{ st Area}(Y) = 1, \; \phi : \mathbb{T}^2 \to Y \text{ diffeo "MARKING"} \Big\} \Big/ \sim$$

$(Y_1, \phi_1) \sim (Y_2, \phi_2)$ IF

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & Y_1 \arrow[d, dashed, "f"] \\ & Y_2 \end{tikzcd}$$

$\exists$ isometry $f : Y_1 \to Y_2$ st $f \circ \phi_1 \simeq \phi_2$ HOMOTOPIC

THM $\quad \mathrm{Teich}_{\mathbb{R}^2}(\mathbb{T}^2) \cong \mathbb{H}^2$

② $M = S_g$ $g \geq 2$, $(G, X) = (PSl_2 \mathbb{R}, \mathbb{H}^2)$

$$\mathrm{Teich}(S_g) := \left\{ (Y, \phi) \;\middle|\; \begin{array}{l} Y = \text{hyp str on } S \\ \phi : \pi_1 S \longrightarrow PSl_2 \mathbb{R} \end{array} \right\} \Big/ \sim$$

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$\exists$ isom. $g : Y_1 \longrightarrow Y_2$

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$\text{Teich}(S_g)$ <u>TEICHMÜLLER SPACE</u> (or FRICKE SPACE)

<u>THM</u> $\text{Teich}(S_g) \overset{1-1}{\longleftrightarrow} \left\{ \rho : \pi_1 S \to PSl_2\mathbb{R} \text{ discrete \& faithful} \right\} \big/ PGl_2\mathbb{R}.$

<u>FUCHS</u>

These representations are called <u>FUCHSIAN reps</u>

Some of the nice features of $\mathrm{Teich}(S_g)$:

① $\mathrm{Teich}(S_g) \underset{\text{diffeo}}{\cong} \mathbb{R}^{6(g-1)}$ (Fenchel-Nielsen coord.)

② $\mathrm{Mod}(S_g) = \mathrm{Homeo}^+(S_g) / \mathrm{Homeo}_0(S_g)$ acts prop. disc. on $\mathrm{Teich}(S_g)$

③ $\exists$ $\mathrm{Mod}(S_g)$-invariant metrics on $\mathrm{Teich}(S_g)$:

- Teichmüller metric
- Weil-Petersson metric
- Thurston asymmetric metric

④ Thurston compactification

⑤ Collar lemma

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Higher Teichmüller theory: can we find c.c. of $\mathrm{Hom}(\Gamma, PSL_d\mathbb{R})/PSL_d\mathbb{R}$ (all made of discrete & faithful reps) which have some of these nice features?

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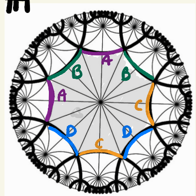
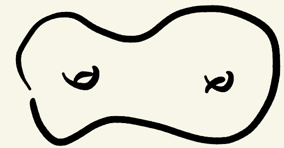
Quasi-Fuchsian reps

Given a Fuchsian rep $\rho : \pi_1 S \longrightarrow PSL_2 \mathbb{R}$

$\rho \curvearrowright \mathbb{H}^2$



$\rightsquigarrow \quad \mathbb{H}^2 / \rho(\pi_1 S) \cong$



S

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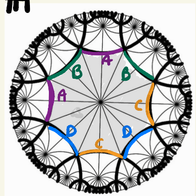
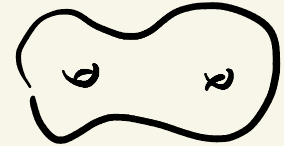


$\rightsquigarrow \mathbb{H}^2/\rho(\pi_1 S) \cong$



S

There is a natural embedding $PSl_2\mathbb{R} \subset PSL_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3)$.

So given $\rho : \pi_1 S \to PSl_2\mathbb{R}$ is Fuchsian,

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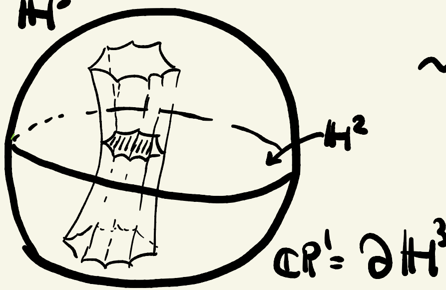

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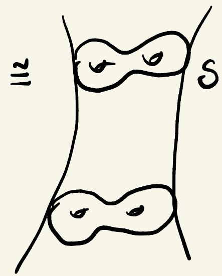

$S \times (-1,1)$

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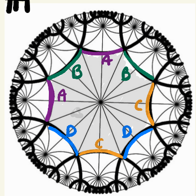

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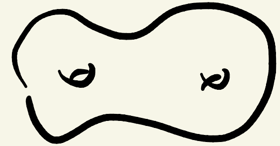

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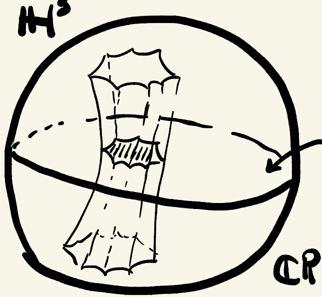
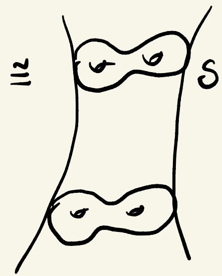


$\mathbb{H}^2$

$\mathbb{C}P^1 = \partial\mathbb{H}^3$

$\rightsquigarrow \mathbb{H}^3/\Gamma_\rho \cong$



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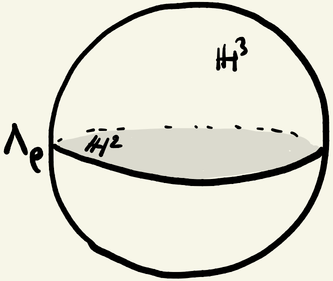
RECALL:

Λ_ρ = LIMIT SET

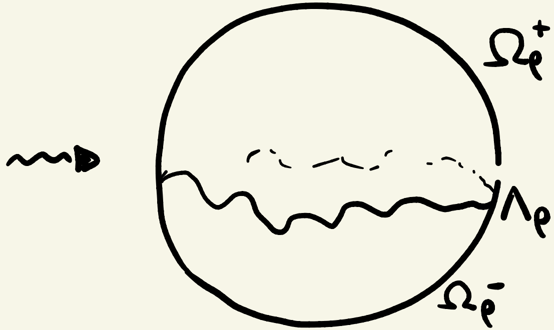
= set of ACCUMULATION points in $\mathbb{C}P^1$.

We can deform this picture (w/ a quasi conformal deformation) & most geom. properties are preserved:

DEF A discrete & faithful rep $\rho: \pi_1 S \to PSl_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3)$ is called **quasi-Fuchsian** if its limit set Λ_ρ is a Jordan curve.



Fuchsian



quasi-Fuchsian

$M_\rho = \mathbb{H}^3/\Gamma_\rho \cong S \times (-1,1)$ hyp 3-mfd.

$\partial M_\rho = \Omega_\rho/\Gamma_\rho \cong S \sqcup S$ (where $\Omega_\rho = \mathbb{CP}^1 \setminus \Lambda_\rho = \Omega_\rho^+ \sqcup \Omega_\rho^-$)
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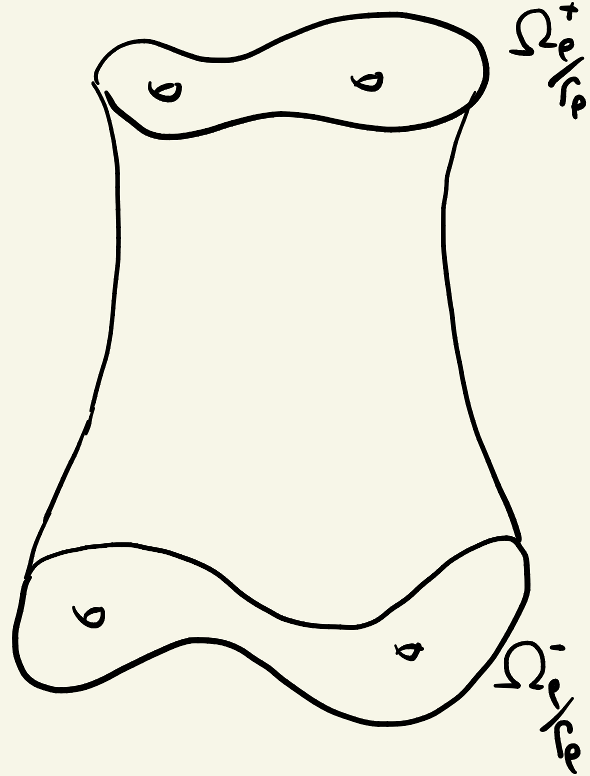
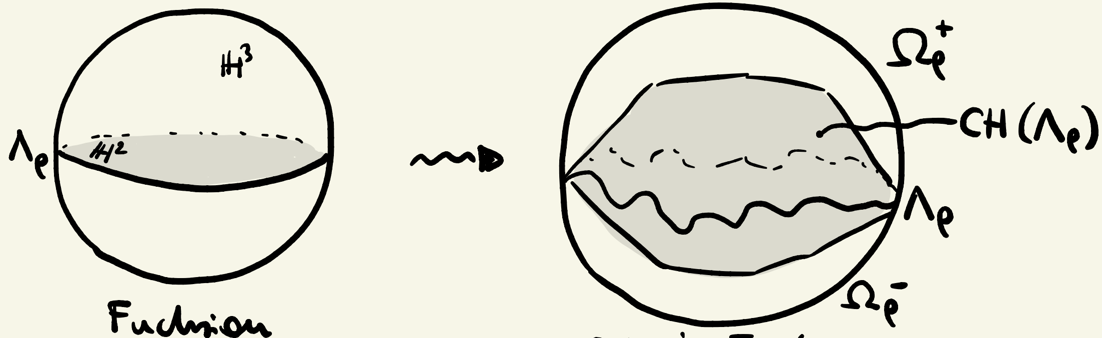
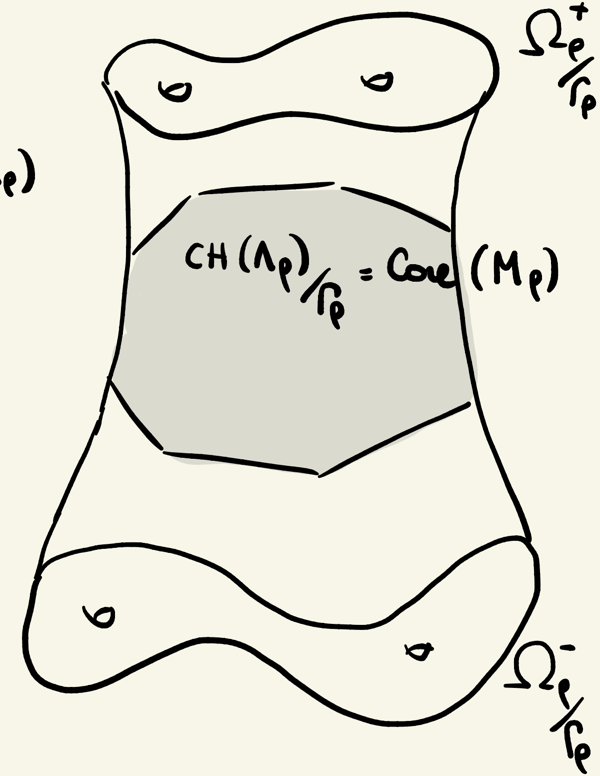




Fuchsian quasi-Fuchsian



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$\overline{M}_\rho = (\mathbb{H}^3 \cup \Omega_\rho)/\Gamma_\rho \cong S \times [-1,1]$

$\mathrm{Core}(M_\rho) = CH(\Lambda_\rho)/\Gamma_\rho \cong S \times [-1,1]$ is the SMALLEST convex subset of M_ρ st

<u>CONVEX CORE</u>

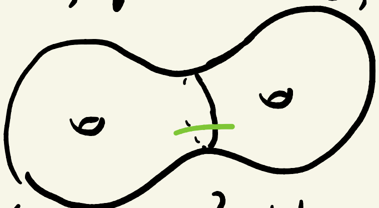
$i : \mathrm{Core}(M_\rho) \hookrightarrow M_\rho$ is a homotopy equivalence.

(EXAMPLES of HYP. 3-mfds & CONVEX COCOMPACT REPS!)

Earthquakes, Cataclysms & Pleated surfaces

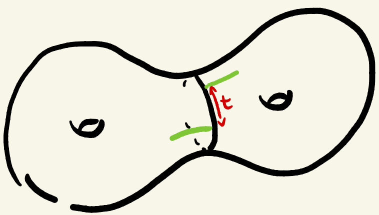
• (Left) **Earthquakes:**

$t \in \mathbb{R}$, γ = s.c.c. in S, $X \in \mathrm{Teich}(S)$

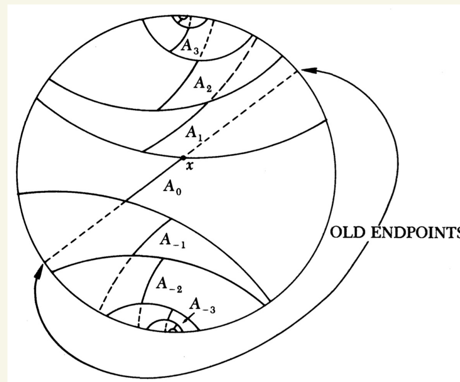


X

$\Big\downarrow \; \mathcal{E}^{l}_{(\gamma,t)}(X)$



i.e. $\tilde{S}$



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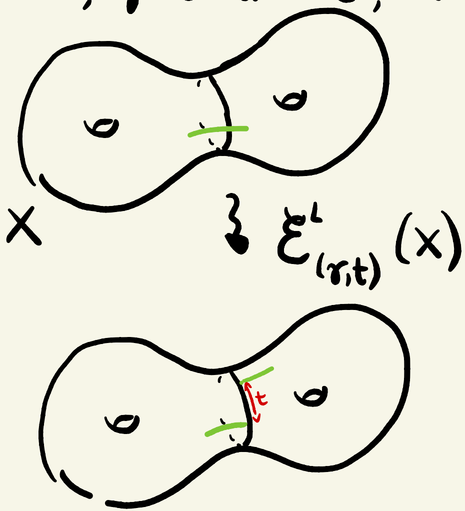

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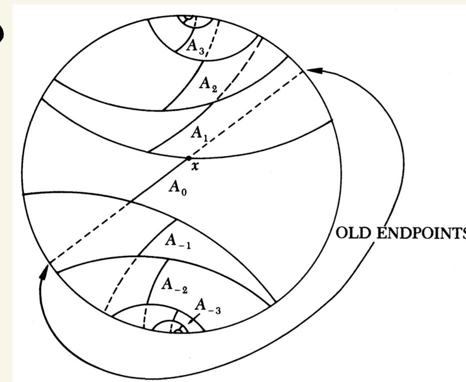
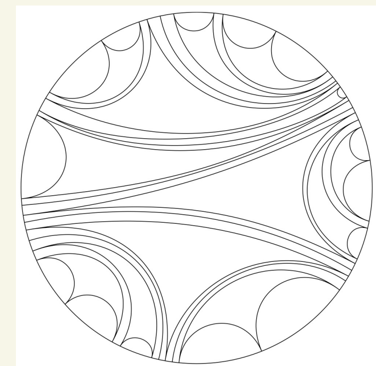


This process can be extend to geodesic laminations



DEF λ = <u>GEODESIC LAMINATION</u> in a hyp surface X is a closed set which is a union of simple geodesics, called LEAVES.

DEF A <u>measure</u> μ on λ is a measure on hyp arcs transverse to λ invariant under push forward along the leaves of λ.

$\boxed{\text{EARTHQUAKE THM}}$ (THURSTON '76; KERCKHOFF '83) $\forall X, Y \in \mathcal{T}eich(S)$

$\exists!\ \lambda \in ML(S)$ s.t. $\mathcal{E}^{L}_{\lambda}(X) = Y$.

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- CATACLYSMS (THURSTON): allow both left & right shear.

  They are parametrised by (SHEARING) COCYCLES $\mathcal{H}(\lambda; \mathbb{R})$

DEF $\alpha \in \mathcal{H}(\lambda; \mathbb{R})$ SHEARING COCYCLE is a map

$\alpha : \{\text{arcs} \pitchfork \lambda\} \to \mathbb{R}$ FINITELY (NOT COUNTABLY) additive & "λ-invariant".

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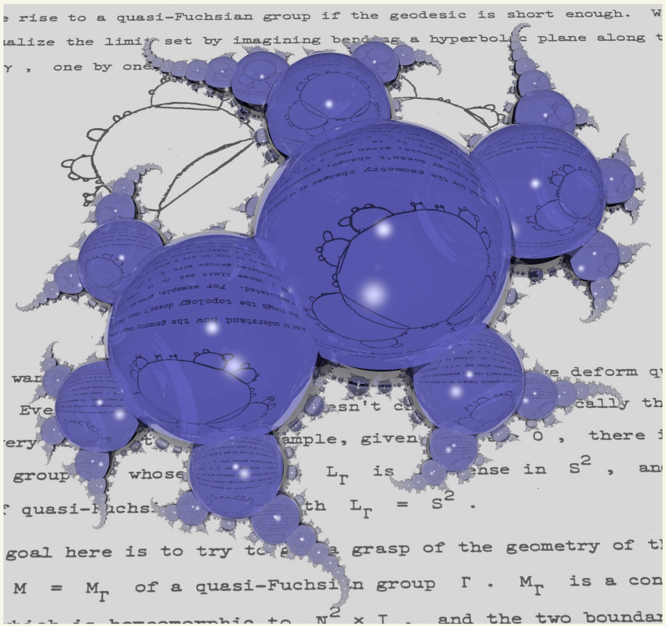
$\boxed{\text{THM (BONAHON)}}$ $\mathrm{Teich}(S) \longrightarrow \mathcal{H}(\lambda; \mathbb{R})$ is $\mathbb{R}$-analytic homeo.
$\rho \longmapsto \sigma_\rho$ = shearing cocycle determined by ρ.
Image $= \mathcal{C}(\lambda)$ is an open convex, bdd by finitely many faces.

PLEATED SURFACES (THURSTON): allow "bending"

<u>DEF</u> (ABSTRACT PLEATED SURFACE) $f=(\tilde{f},\rho)$ w/ $\rho:\pi_1 S\to PSl_2\mathbb{C}$ homeo,
$\tilde{f}:\tilde{S}\to\mathbb{H}^3$ ρ-equiv. & s.t.

- $\tilde{f}$: {leaves of $\tilde{\lambda}$} $\mapsto$ {geods in $\mathbb{H}^3$}
- $\tilde{f}$: {$\tilde{S}\setminus\lambda$} $\longmapsto$ {tot. geod. pieces}
- pull-back $f^*\rho_{hyp}$ of hyp metric is hyp. metric on $\tilde{S}$.



<u>PIC</u>: BROCK-DUMAS

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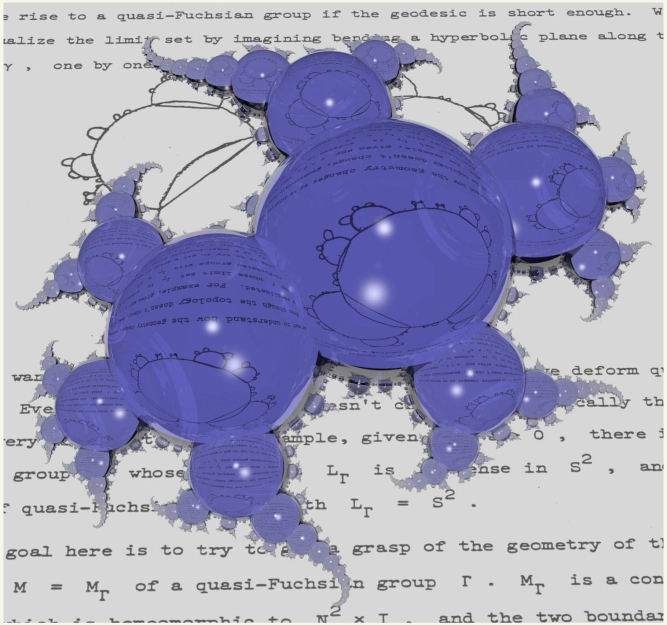

<u>PIC</u>: BROCK - DUMAS

<u>RK</u> λ MAXIMAL $\Rightarrow f$! det by ρ

$R(\lambda) := \{[\rho] \mid \exists\, f = (\tilde{f}, \rho) \text{ pl. surface, pleated along } \lambda\}$

<u>THM (BONAHON)</u> $R(\lambda) \longrightarrow \mathcal{C}(\lambda) \times \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$

$\rho \longmapsto (\sigma_\rho, \beta_\rho)$

SHEAR - BEND COCYCLES!

is a biholom. homeo.

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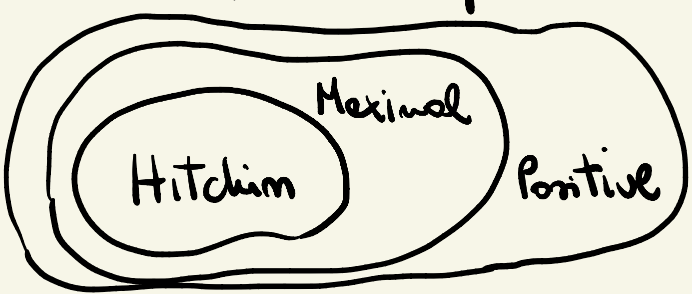
Higher Teichmüller theory:

DEF A <u>HIGHER TEICHMÜLLER SPACE</u> is a c.c. of $\mathrm{Hom}(\Gamma, G)/G$ all made of discrete & faithful reps which have some of the nice features of $\mathrm{Teich}(S)$.

$\boxed{Q}$: Can we find such spaces?

$\boxed{A}$: Yes! HITCHIN reps (HITCHIN; LABOURIE; FOCK-GONCHAROV)
MAXIMAL reps (BURGER-IOZZI-WIENHARD; BIW-LABOURIE)
POSITIVE reps (FOCK-GONCHAROV; GUICHARD-LABOURIE-WIENHARD)



HITCHIN REPS

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• $\rho : \pi_1 S \to PSL_d\mathbb{R}$ is HITCHIN if it can be continuosly deformed to a Fuchsian rep

$Hit_d(S) = \{\rho : \pi_1 S \to PSL_d\mathbb{R} \text{ Hitchin}\}$ is an example of HIGHER TEICHMÜLLER COMPONENT.

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Some "Teichmüller features":

① (HITCHIN 1992; BONAHON-DREYER 2014) $Hit_d(S) \cong \mathbb{R}^{2(g-1)\boxed{(d^2-1)}}$ $\dim(PSL_d\mathbb{R})$

② (LABOURIE 2008) $Mod(S) \curvearrowright Hit_d(S)$ prop. discont.

③ (PARREAU 2012) Compactification of $Hit_d(S)$

④ • (BURGER 1995; GUERITAUD-KASSEL 2013)
Generalization of THURSTON metric (asymm.)
• (BRIDGEMAN-CANARY-LABOURIE-SAMBARINO 2015; LI 2016)
Generalization of Weil-Petersson metric

⑤ (LEE-ZHANG 2017) Collar lemma

On the other hand...

$$QF(S) = \{\rho : \pi_1 S \to PSL_2\mathbb{C} \text{ quasi-Fuchsian}\}/PSL_2\mathbb{C} \underset{\text{open}}{\subset} \mathcal{X}(\pi_1 S, PSL_2\mathbb{C})$$

is <u>not</u> a c.c. of $\mathcal{X}(\pi_1 S, PSl_2\mathbb{C})$, so it is $\boxed{\text{NOT}}$ a HIGHER TEICHMÜLLER SPACE

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<u>DEF</u> $\rho : \pi_1 S \longrightarrow PGL_d\mathbb{R}$ is **k-ANOSOV** if $\exists$ ρ-equivariant

$$\xi_\rho : \partial_\infty(\pi_1 S) = \mathbb{RP}^1 \longrightarrow \mathcal{F}_{k,d-k}(\mathbb{R}^d)$$ which is

- <u>cts</u>
- <u>TRANSVERSE</u> ($: \forall x \neq x' \in \partial_\infty(\pi_1 S)$ $\xi_\rho^{(k)}(x)$, $\xi_\rho^{(d-k)}(x')$ are transverse)
- it satisfies a <u>UNIFORM CONTRACTION/EXPANSION</u> on $E^\rho = \Gamma \backslash T^1\tilde{S} \times \mathbb{R}^d$

  ξ_ρ induces a decomposition of $E^\rho = \Theta^\rho + \Xi^\rho$

  invariant under the flow $\downarrow$

  $T^1 S = \Gamma \backslash T^1 \tilde{S}$

the flow uniformly contracts Θ^ρ wrt Ξ^ρ

$(\Gamma\text{-action} : \gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \rho(\gamma)\cdot v))$

On the other hand...

$QF(S) = \{\rho : \pi_1 S \to PSL_2\mathbb{C} \text{ quasi-Fuchsian}\} / PSL_2\mathbb{C} \subset_{\text{open}} \mathcal{X}(\pi_1 S, PSL_2\mathbb{C})$

is <u>not</u> a c.c. of $\mathcal{X}(\pi_1 S, PSL_2\mathbb{C})$, so it is $\boxed{\text{NOT}}$ a HIGHER TEICHMÜLLER SPACE but it still has a lot of nice features & some new/different ones too.

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$$\downarrow$$

$$T^1 S = \Gamma \backslash T^1\tilde{S}$$

$(\Gamma\text{-action}: \ \gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot v))$

<u>THM (LABOURIE)</u> $\rho \in Hit_d(S) \Rightarrow \rho$ is k-ANOSOV $\forall k = 1..d-1$ [BOREL-ANOSOV)

So $\xi : \partial(\pi_1 S) \longrightarrow \mathcal{F}(\mathbb{R}^d)$

1. (G,X) - STRUCTURES & Teichmüller space

2. QUASI-FUCHSIAN MANIFOLDS & PLEATED SURFACES

3. Higher Teichmuller spaces & Anosov representations

4. Few new results

QUESTIONS for TODAY:

Q ① Is there an interpretation of these conn. comp. of Anosov reps as deformation spaces of geometric structures?

Q ② Is there a "higher rank" definition of pleated surface? If yes, can you describe their parametrization?

A $\boxed{\text{YES!}}$:

① joint w/ ALESSANDRINI - THOLOZAN - WIENHARD

② joint w/ MARTONE - MAZZOLI - ZHANG

TOPOLOGY OF QUOTIENTS of DOMAINS of DISCONTINUITY

GUICHARD-WIENHARD; KAPOVICH-LEEB-PORTI described COCOMPACT domains of discontinuity $\Omega_\rho \subset Gr_{k'}(\mathbb{R}^d)$ for k-Anosov reps $\rho : \Gamma \longrightarrow PGL_d\mathbb{R}$

$\mathcal{X}_k(\pi_1 S, PGL_d\mathbb{R}) = \{\rho \mid k\text{-Anosov}\}$

THM 1 (ALESSANDRINI-M-THOLOZAN-WIENHARD)

Let C be a c.c. in $\mathcal{X}_k(\pi_1 S, PGL_d\mathbb{R})$ containing a Fuchsian rep j.
$\forall \rho \in C$, $\forall$ non-empty cocompact d.o.d. $\Omega_\rho \subset Gr_{k'}(\mathbb{R}^d)$

$\Rightarrow$ $\Gamma_\rho \backslash \Omega_\rho$ is a smooth fiber bundle over Σ.

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Let C be a c.c. in $\mathcal{H}_k(\pi_1 S, PGL_d\mathbb{R})$ containing a Fuchsian rep j.
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$\Rightarrow \Gamma_\rho \backslash \Omega_\rho$ is a smooth fiber bundle over Σ.

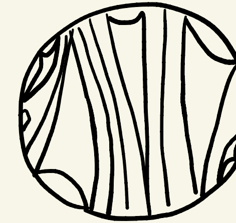
RK In general, it is hard to understand the topology of F.

RK Thm 1 was understood in some specific examples before.

THM 2 (A-M-T-W) If $G = Sp_4\mathbb{C}$ & $P = Stab_G(\ell)$, $\ell \in \mathbb{CP}^3$, then $F \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

PLEATED SURFACES for $\mathrm{Hom}(\pi_1 S, PSL_d \mathbb{C})$

λ = MAXIMAL geod. lam. in S



DEF (λ, k)-ANOSOV reps

$\rho : \pi_1 S \to PGL_d \mathbb{C}$ is (λ, k)-ANOSOV IF

$\exists$ ρ-equiv. $\xi : \partial\tilde{\lambda} \to \mathcal{F}_{k,d-k}(\mathbb{C}^d)$ (locally Hölder) cts)

- λ-cts $\left(\xi_{\tilde{\lambda}} : \tilde{\Lambda}^{or} \to (\mathcal{F}_{k,d-k}(\mathbb{C}^d))^2\right.$ is
- λ-Transverse $\left(\forall\, g \in \tilde{\Lambda}^{\circ} \quad \xi^{(k)}(g^+) + \xi^{(d-k)}(g^-) = \mathbb{C}^d\right)$
- uniform contraction/expansion conditions on $E^{\lambda}_{\rho} := {}_{\Gamma}\backslash T^1\tilde{\lambda} \times \mathbb{C}^d$

$$\begin{array}{c} E^{\lambda}_{\rho} \\ \downarrow \\ T^1\lambda = {}_{\Gamma}\backslash T^1\tilde{\lambda} \end{array}$$

PLEATED SURFACES for $\mathrm{Hom}(\pi_1 S, PSL_d\mathbb{C})$

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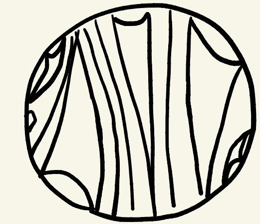

DEF ((λ,k)-Anosov reps) $\rho:\pi_1 S \to PGL_d\mathbb{C}$ is (λ,k)-Anosov if

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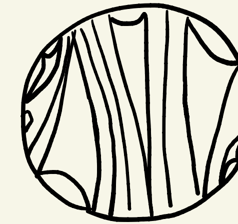
DEF (d-PLEATED SURFACES) (ρ, ξ_ρ) d-PLEATED SURFACE if

- $\rho:\pi_1 S \to PGL_d\mathbb{C}$ is λ-Borel Anosov ($:\Leftrightarrow$ (λ,k)-Anosov $\forall k = 1 \dots d-1$)
- $\xi_\rho: \partial\tilde{\lambda} \to \mathcal{F}(\mathbb{C}^d)$ is λ-generic ($:\Leftrightarrow$ $\forall$ triangle $T \in \tilde{\Delta}^\circ$ $\xi(T)$ is in general position)

$$R_d(\lambda) := \{\rho \mid (\rho, \xi_\rho) \ d\text{-pleated surfaces}\} \subset \mathrm{Hom}(\pi_1 S, PGL_d\mathbb{C})$$

PLEATED SURFACES for $\mathrm{Hom}(\pi_1 S, \mathrm{PSL}_d\,\mathbb{C})$

λ = MAXIMAL geod. lam. in S



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DEF (d-PLEATED SURFACES) (ρ, ξ_ρ) d-PLEATED SURFACE IF

- $\rho : \pi_1 S \to \mathrm{PGL}_d\,\mathbb{C}$ is BOREL ANOSOV ($:\Leftrightarrow$ (λ,k)-Anosov $\forall\, k = 1 \ldots d-1$)
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$$\mathcal{R}_d(\lambda) := \{\rho \mid (\rho, \xi_\rho)\ d\text{-pleated SURFACES}\} \subset \mathrm{Hom}(\pi_1 S, \mathrm{PGL}_d\,\mathbb{C})$$

THM 3 (M-MARTONE-MAZZOLI-ZHANG)

$$\phi^\lambda : \mathcal{R}_d(\lambda) \longrightarrow \mathcal{C}_d(\lambda) \times \mathcal{Y}_d(\lambda; \mathbb{R}/2\pi\mathbb{Z}) \times \mathcal{N}_d$$ is a biholomorphism.

Thank you !!!