The symplectic geometry of connections

[Katrin Wehrheim - they/them]

* pronouns & other microaggressions

FAQ: What if someone gets it* wrong?

interrupt - apologize & correct - move on & learn

Why might symplectic geometry reproduce gauge theoretic invariants for 3- and 4-manifolds ?

... and why might one care?

- me
- you
- math community
- humanity

For background, citations, and speculations see Floer Field Philosophy.

Board: https://bit.ly/KWboard

Q: What is a symplectic manifold ?

A: A manifold X with symplectic structure ω (generalizing position-momentum pairing):

ω nondegenerate closed 2-form

$\exists$ almost complex structure $J: TX \to TX$, $J^2 = -id$ s.t. $\omega(\cdot, J\cdot)$ is a metric

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Taubes SW=Gr theorem: (X,ω) symplectic, $\dim X = 4$

$$\Rightarrow \forall \beta \in H_2(X) \quad SW(X, \mathfrak{s}_{(\omega,J)} + \beta) = Gr^{\text{Taubes}}(X, \omega, \beta)$$

$$\# \frac{\{(A,\psi) : D_A\psi = 0,\ F_A^+ = \tfrac{1}{2}\rho(\psi) - i\eta\}}{\text{gauge equivalence}} \qquad \#^T \frac{\left\{ \begin{array}{l} u: (\Sigma, j) \to (X, J): \\ du \circ j = J(u) \circ du,\ u_*[\Sigma] = \beta \end{array} \right\}^{Gr}}{\text{reparametrization}}$$

Gromov-compactified moduli space of pseudoholomorphic curves

why? for $\eta = r\omega$, $r \to \infty$, zeros of part of $\psi \approx u(\Sigma)$

Why? WHY? WTFY?

What do you notice ? What do you wonder ?

Why does
This
Further
Yanna's
cause?

My accountability offering: ANTIRACISM PEDAGOGY practice group

- listening/reading tour
- transform our teaching
- feedback & support
- accessible & sustainable
  - → international zoom option
  - → in-person with lunch or dinner?

0th meeting for
time & format search
FRIDAY 9/2 1-2pm
1st floor lunch deck
and/or email
katrin@ math.berkeley.edu

Antiracism 101 Sept.6-Nov.17 by SURJ-SF

Atiyah-Floer Conjecture: Y 3-manifold

$\|$

Heegård decomposition $H_0 \cup_\Sigma H_1$ into handle bodies H_0, H_1
w. boundary $\partial H_0 = \Sigma = \partial H_1$

$\Downarrow$

instanton Floer homology of Y = Floer homology of symplectic data arising from flat conn. on Σ, H_0, H_1

💩 🤢 usually neither is defined

🔮 😻 use it as general guidance

Heegard-Floer theory: 3-manifold invariants obtained by

- choice of decomposition $Y = H_0 \cup_\Sigma H_1$ into handle bodies H_0, H_1
- dimensional reduction of Seiberg-Witten $\Sigma \rightsquigarrow M_\Sigma = \mathrm{Sym}^g \Sigma$
- count of pseudohol. curves in M_Σ with boundary on L_{H_0}, L_{H_1}

$$\begin{array}{ccc} \Sigma & \rightsquigarrow & M_\Sigma = \mathrm{Sym}^g\Sigma \\ & & \cup \\ H & \rightsquigarrow & L_H \text{ Lagrangian submanifold} \\ \partial H = \Sigma & & \end{array}$$

- some algebra

Thm: Heegård Floer theory of $L_{H_0}, L_{H_1} \subset M_\Sigma$ is independent (up to...algebra...) of choice of Heegård decomposition $Y = H_0 \cup_\Sigma H_1$

$\cong H_0' \cup_{\Sigma'} H_1'$

Why? [Ozsváth-Szabó]

Why? Conjecture [W-Woodward, Perutz]: $\Sigma \rightsquigarrow \mathrm{Sym}^g\Sigma$, $H \rightsquigarrow L_H$ is part of a functor $\mathrm{Bor}_{2+1} \rightsquigarrow \mathrm{Symp}$

$$Y = \begin{array}{c} \emptyset \xrightarrow{H_0} \Sigma \xrightarrow{H_1} \emptyset \\ \text{diffeo} \\ \emptyset \xrightarrow{H_0'} \Sigma' \xrightarrow{H_1'} \emptyset \end{array} \quad \rightsquigarrow \quad \begin{array}{c} pt \xrightarrow{L_{H_0}} M_\Sigma \xrightarrow{L_{H_1}} pt \\ \text{same HF} \\ pt \xrightarrow{L_{H_0'}} M_{\Sigma'} \xrightarrow{L_{H_1'}} pt \end{array}$$

WHY? Thm [WW]: Floer homology is "reasonable" in Symp

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general construction principle for topological invariants

	$\underline{\mathrm{Bor}_{2+1}}$ $\xrightarrow{\text{functor}}$	$\underline{\mathrm{Symp}}$	from gauge theory
objects:	closed 2-manifolds	symplectic manifolds	
morphisms:	3-dim. cobordisms	Lagrangian relations	
composition:			

Foundational Example: connections on trivial G-bundles

$\underline{\mathrm{Bor}_{2+1}} \xrightarrow{\text{functor}} \underline{\mathrm{Symp}}$ $\qquad \mathfrak{g} = \mathrm{Lie}\, G$

objects: closed 2-manifolds $\qquad$ symplectic manifolds

 $\Sigma \longmapsto \left(\mathcal{A}(\Sigma) = \Omega^1(\Sigma; \mathfrak{g}),\ \boxed{\omega(\alpha,\beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle}\right)$

pick Riem. metric

$\rightsquigarrow * : \Omega^1 \to \Omega^1$

$*^2 = -id$

L^2-metric $= \int \langle \alpha \wedge * \beta \rangle$

- ✓ bilinear
- ✓ antisymmetric
- closed = constant
- nondegenerate

morphisms: 3-dim. cobordisms $\qquad$ Lagrangian relations

Σ_1  Σ_2 $\longmapsto \mathcal{L}_Y = \{ (A|_{\Sigma_1}, A|_{\Sigma_2}) \mid \boxed{A \in \mathcal{A}(Y),\ F_A = 0} \}$

$\subset \big(\underbrace{\mathcal{A}(\Sigma_1) \times \mathcal{A}(\Sigma_2)}_{\mathcal{A}(\partial Y)},\ \underbrace{-\omega_{\Sigma_1} \oplus \omega_{\Sigma_2}}_{\omega} \big)$

Lagrangian: $\omega|_{T\mathcal{L}} \equiv 0, \dots$

$\alpha, \beta \in T_{A|_{\partial Y}} \mathcal{L}_Y \Rightarrow \omega(\alpha,\beta) = \int_{\partial Y} \langle \alpha \wedge \beta \rangle$

$\overset{\shortparallel}{(\ker d_A)|_{\partial Y}}$

$= \int_Y d \langle \tilde{\alpha} \wedge \tilde{\beta} \rangle = \int_Y \langle \underbrace{d_A \tilde{\alpha}}_{=0} \wedge \tilde{\beta} \rangle - \langle \tilde{\alpha} \wedge \underbrace{d_A \tilde{\beta}}_{=0} \rangle = 0$

composition:  $\qquad \mathcal{L}_{Y \cup Z} = \mathcal{L}_Y \circ \mathcal{L}_Z$

Foundational Example: connections on trivial G-bundles

$$\underline{\mathrm{Bor}_{2+1}} \xrightarrow{\text{partial functor}} \underline{\mathrm{Symp}}$$

objects: closed 2-manifolds — symplectic manifolds

$$\Sigma \longmapsto \left(\mathcal{A}(\Sigma)=\Omega^1(\Sigma;\mathfrak{g}),\ \omega(\alpha,\beta)=\int_\Sigma \langle \alpha\wedge\beta\rangle\right)$$

with complex structures $J = *$
from metrics on Σ

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morphisms: simple 3-dim. cobordisms — Lagrangian relations

$T\mathcal{L}$ needs to be $\mathrm{im}\, d_A$ + half of H^1

$$\longmapsto \mathcal{L}_Y = \{(A|_{\Sigma_1}, A|_{\Sigma_2}) \mid A\in\mathcal{A}(Y),\ F_A = 0\}$$

$$\subset (\mathcal{A}(\Sigma_1)\times\mathcal{A}(\Sigma_2),\ -\omega_1\oplus\omega_2) = (M,\omega)$$

Hodge decomp.: $\Omega^1(\partial Y;\mathfrak{g}) =$

$$\underbrace{\mathrm{im}\, d_{A|_{\partial Y}}}_{\cap\, T\mathcal{L}} \oplus \underbrace{\mathrm{im}\, * d_{A|_{\partial Y}}}_{\cap\, *T\mathcal{L}} \oplus \underbrace{\ker(d_{A|_{\partial Y}}, d^*_{A|_{\partial Y}})}_{H^1_A(\partial Y)}$$

Lagrangian: ✓ $\omega|_{T\mathcal{L}} \equiv 0$
✓ $\boxed{T\mathcal{L} \oplus JT\mathcal{L} = TM}$

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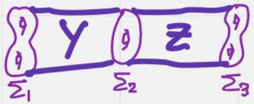
composition:



$$\mathcal{A}(\Sigma_1) \xrightarrow{\mathcal{L}_Y} \mathcal{A}(\Sigma_2) \xrightarrow{\mathcal{L}_Z} \mathcal{A}(\Sigma_3)$$

$$\mathcal{L}_{Y\cup Z} = \{(A_1, A_3) \mid \exists A_2 : (A_1,A_2)\in\mathcal{L}_Y,\ (A_2,A_3)\in\mathcal{L}_Z\}$$

$$=: \mathcal{L}_Y \circ \mathcal{L}_Z$$ composition of general relations

Atiyah-Floer Example: connections on trivial G-bundles // gauge

$$\mathrm{Bor}_{2+1} \xrightarrow{\text{functor}} \mathrm{Symp}$$

objects: closed 2-manifolds — symplectic manifolds

 $\Sigma \longmapsto M_\Sigma := \mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma) = \dfrac{\text{flat conn.}}{\text{gauge}} = \dfrac{\mathrm{Hom}(\pi_1(\Sigma), G)}{G}$ " $\dfrac{\mu^{-1}(0)}{\mathcal{G}}$ "

gauge action $\mathcal{G}\times\mathcal{A} \longrightarrow \mathcal{A}$, $(u, A) \mapsto u^{-1}Au + u^{-1}du$ is "Hamiltonian with moment map" $\boxed{\mu(A) = *F_A}$

💩 🤢 $\mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma) \simeq \dfrac{\{(a_1, b_1 \ldots a_g, b_g) \in G^{2g} \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1\}}{\text{conjugation with } G}$ is very singular!

🔮 😻 ...but on smooth part inherits symplectic structure & Lagrangians ☡

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morphisms: 3-dim. cobordisms — Lagrangian relations

Σ_1  Σ_2 $\longmapsto L_Y = \{(A|_{\Sigma_1}, A|_{\Sigma_2}) \mid A \in \mathcal{A}(Y), F_A = 0\} \,/\, \mathcal{G}(\Sigma_1)\times\mathcal{G}(\Sigma_2)$

$$\Big\{([g_1],[g_2]) \in M_{\Sigma_1}\times M_{\Sigma_2} \;\Big|\; \exists g : \pi_1(Y) \to G \Big\} =$$

where $g_2 : \pi_1(\Sigma_2) \to \pi_1(Y) \to G$ and $g_1 : \pi_1(\Sigma_1) \to \pi_1(Y) \to G$.

functoriality
$L_{Y\cup Z} = L_Y \circ L_Z$

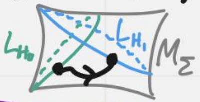
Atiyah-Floer Conjecture: Y 3-manifold, $H_*(Y;\mathbb{Z}) \simeq H_*(S^3;\mathbb{Z})$

$$Y = H_0 \cup_\Sigma H_1 \quad \text{handle bodies}$$

instanton Floer homology of Y = Floer homology of $L_{H_0}, L_{H_1} \subset M_\Sigma$

"Morse homology of $cs : \mathcal{A}(Y)/\mathcal{G} \to \mathbb{R}/4\pi\mathbb{Z}$"

"Morse homology of symplectic action"

$\{$paths  $\} \to \mathbb{R}/$ SMALL PRINT

why? / Why? talk didn't get here

TFT-ish: ASD reduction induces functor $Bor_{2+1} \to Symp$

Bold Plan: Compare Donaldson & symplectic 2-functors on $Bor_{2+1+\varepsilon}$

Adiabatic Limit [Salamon]: $A(s,t) + \Phi\, ds + \Psi\, dt$ is ASD on $\times\, \Sigma$

$ds^2 + dt^2 + \varepsilon g_\Sigma$

$\Updownarrow$

fixed energy
$$\int |\partial_s A - d_A \Phi|^2 + \varepsilon^{-2} |F_A|^2$$

$$(\partial_s A - d_A \Phi) + *_\Sigma(\partial_t A - d_A \Psi) = 0$$
$$\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \varepsilon^{-2} F_A = 0$$

$\varepsilon \to 0$ limit

$A : \to \mathcal{A}(\Sigma)$ solves

$$\partial_s [A] + *_\Sigma \partial_t [A] = 0$$
$$F_A = 0$$

3&4-dim "quilted" Atiyah-Floer conjecture: ASD moduli spaces and Atiyah-Floer + pseudoholomorphic moduli spaces induce isomorphic 2-functors $\mathrm{Bor}_{2+1+1} \to \mathcal{C}at$

$\mathcal{C}at$
categories
functors
nat. transf.

$$\begin{tikzcd}
 & \mathcal{C}at & \\
\mathrm{DonBor} \arrow[ur, bend left, "\text{Yoneda}"] & \widetilde{\mathrm{Bor}}_{2+1+1} \arrow[l, "\text{ASD}"'] \arrow[r, "\text{Atiyah-Floer}"] & \mathrm{Symp}^{G} \arrow[ul, bend right, "\text{Yoneda}"']
\end{tikzcd}$$

DonBor	Bor_{2+1+1}	Symp^G
2-manifolds	2-manifolds	symplectic quotients of G-representations
3-cobordisms	3-cobordisms with Morse functions	composable chains of handle attachment Lagrangians
instanton Floer classes	4-cobordisms with Morse 2-functions	symplectic Floer classes

Lurie-TFT style proof idea:

- put enough structure on all 2-categories
- show functors respect structure
- compare on generators:
  - 2d surfaces
  - 3d handle attachments