Mirror Symmetry For Higgs Bundles, Generalized Hyperpolygons and More

Laura Schaposnik – University of Illinois at Chicago

THE PLAN

1. An overview of directions

2. Integrable systems + Mirror symmetry

3. Generalized Hyperpolygons

4. Other projects

HIGGS BUNDLES

→ ASSOCIATED TO A COMPLEX CURVE Σ
AND A REDUCTIVE COMPLEX LIE GROUP G

Hitchin '87
2d REDUCTION OF YANG-MILLS EQUATIONS

HIGGS BUNDLES

→ ASSOCIATED TO A COMPLEX CURVE Σ
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Hitchin '87
2d REDUCTION OF YANG-MILLS EQUATIONS

$$\bar{\partial}_A \Phi = 0$$

$$F_A + [\Phi, \Phi^*] = 0$$

HIGGS BUNDLES

ASSOCIATED TO A COMPLEX CURVE Σ AND A REDUCTIVE COMPLEX LIE GROUP G

Hitchin '87

2d REDUCTION OF YANG-MILLS EQUATIONS

$$\bar{\partial}_A \Phi = 0$$

$\bar{\partial}$-CONNECTION ON A PRINCIPAL G-BUNDLE P DEFINING HOLOMORPHIC STRUCTURE

$$F_A + [\Phi, \Phi^*] = 0$$

CURVATURE OF UNITARY CONNECTION

$$\nabla_A = \partial_A + \bar{\partial}_A$$

$$\Omega^{1,0}(\Sigma, \mathrm{Ad}P)$$

HIGGS BUNDLES

ASSOCIATED TO A COMPLEX CURVE Σ
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$\bar{\partial}$-CONNECTION ON
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$$\bar{\partial}_A \Phi = 0$$

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CURVATURE OF
UNITARY CONNECTION

$$\nabla_A = \partial_A + \bar{\partial}_A$$

$$\Omega^{1,0}(\Sigma, \mathrm{Ad}P)$$

A SOLUTION $(\bar{\partial}_A, \Phi)$ DETERMINES A FLAT G-CONNECTION

$$\nabla = \nabla_A + \Phi + \Phi^*$$

LEADS TO
HIGGS BUNDLES
CORRESPONDENCES

HIGGS BUNDLES

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

$(\bar{\partial}_A, \Phi)$ ON A COMPACT RIEMANN SURFACE Σ

HIGGS BUNDLES

COMPLEX REDUCTIVE LIE GROUP

$\mathcal{M}_G$ MODULI SPACE OF G-HIGGS BUNDLES

+ STABILITY

$(\bar{\partial}_A, \Phi)$ ON A COMPACT RIEMANN SURFACE Σ

$g \geqslant 2$

HIGGS BUNDLES

COMPLEX REDUCTIVE LIE GROUP

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

+ STABILITY

(E, Φ) ON A COMPACT RIEMANN SURFACE Σ

VECTOR BUNDLE

THE HIGGS FIELD

$\Phi : E \longrightarrow E \otimes T^*\Sigma$

$g \geq 2$

HIGGS BUNDLES

$\mathcal{M}_G$ → MODULI SPACE OF ‸ G-HIGGS BUNDLES

+ STABILITY

(E, Φ) ON A COMPACT RIEMANN SURFACE Σ

↓

QUASI PROJECTIVE SCHEME

SMOOTH UNDER CERTAIN CONDITIONS

HYPERKÄHLER STRUCTURE

INTEGRABLE SYSTEM

HIGGS BUNDLES

COMPLEX REDUCTIVE LIE GROUP

$\mathcal{M}_G \rightarrow$ MODULI SPACE OF G-HIGGS BUNDLES

\+ STABILITY

(E, Φ) ON A COMPACT RIEMANN SURFACE Σ

VECTOR BUNDLE

THE HIGGS FIELD

$\Phi : E \longrightarrow E \otimes T^*\Sigma$

$g \geqslant 2$

EXAMPLE (RANK 2) LET $T^*\Sigma = K$, AND FIX $K^{1/2}$.

FOR $E = K^{1/2} \oplus K^{-1/2}$, CONSIDER

$$\Phi : \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} : (K^{1/2} \oplus K^{-1/2}) \longrightarrow (K^{1/2} \oplus K^{-1/2}) \otimes K$$

HIGGS BUNDLES

COMPLEX REDUCTIVE LIE GROUP

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

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$$\Phi : \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} : \overbrace{(K^{1/2} \oplus K^{-1/2}) \longrightarrow (K^{1/2} \oplus K^{-1/2})}^{\omega \in H^0(\Sigma, K^2)} \otimes K$$

A FAMILY OF HIGGS BUNDLES PARAMETRIZED BY QUADRATIC DIFFERENTIALS

(THE HITCHIN COMPONENT / TEICHMÜLLER SPACE)

THE INTEGRABLE SYSTEM

$$\begin{tikzcd}
\mathcal{M}_G \arrow[d, "h"', ""{name=L}] & & \mathcal{M}_{\mathcal{N}} \arrow[d, ""'{name=R}] \\
\mathcal{A}_G & & \mathcal{A}_{\mathcal{N}}
\arrow[from=L, to=R, leftrightarrow, bend right, shorten=6pt]
\end{tikzcd}$$

WHAT DO CORRESPONDENCES BETWEEN DIFFERENTIALS TELL US ABOUT THE MODULI SPACES?

THE INTEGRABLE SYSTEM

$$\begin{tikzcd}
\mathcal{M}_G \arrow[d, "h"', ""{name=L}] & \mathcal{M}_{^LG} \arrow[d, ""{name=R}] \\
\mathcal{A}_G \arrow[r, phantom, "\cong"] & \mathcal{A}_{^LG}
\arrow[from=L, to=R, leftrightarrow, bend left, "SYZ"]
\end{tikzcd}$$

MIRROR SYMMETRY

(DUALITY BETWEEN CALABI-YAU MANIFOLDS)

THE MODULI SPACES ARE
MIRROR PARTNERS

THE FIBRES ARE DUAL
ABELIAN VARIETIES

HAUSEL-THADDEUS
KAPUSTIN-WITTEN
DONAGI-PANTEV
HITCHIN

AND MORE GENERIC LANGLANDS BUT
VIA HITCHIN SYSTEMS
} ARINKIN, CHAU, DIACONESCU
GUKOV, KATZARKOV, ...

THE INTEGRABLE SYSTEM

$$\begin{tikzcd}
\mathcal{M}_G \arrow[d, "h"', ""{name=L}] & \mathcal{M}_{{}^LG} \arrow[d, ""{name=R}] \\
\mathcal{A}_G \arrow[r, phantom, "\cong"] & \mathcal{A}_{{}^LG}
\arrow[from=L, to=R, leftrightarrow, bend left, "\mathrm{SYZ}"]
\end{tikzcd}$$

MIRROR SYMMETRY
(DUALITY BETWEEN CALABI-YAU MANIFOLDS)

THE MODULI SPACES ARE
MIRROR PARTNERS

THE FIBRES GENERIC ARE DUAL
ABELIAN VARIETIES

WHAT ABOUT THE NON-REGULAR
FIBRES AND THEIR
$\underbrace{\text{SUBSPACES?}}_{\text{BRANES}}$

BRANES & MIRROR SYMMETRY

BRANES OF TYPE (A,B,A), (B,A,A)

(A,A,B), (B,B,B)

W/ BARAGIA

CONSTRUCTED FAMILIES BY IMPOSING SYMMETRIES

BRANES & MIRROR SYMMETRY

$$\begin{tikzcd}
\mathcal{M}_G \arrow[d] & \\
\mathcal{A}_G & \mathcal{M}_{^LG} \arrow[l]
\end{tikzcd}$$

KNOWN: DUAL TYPES

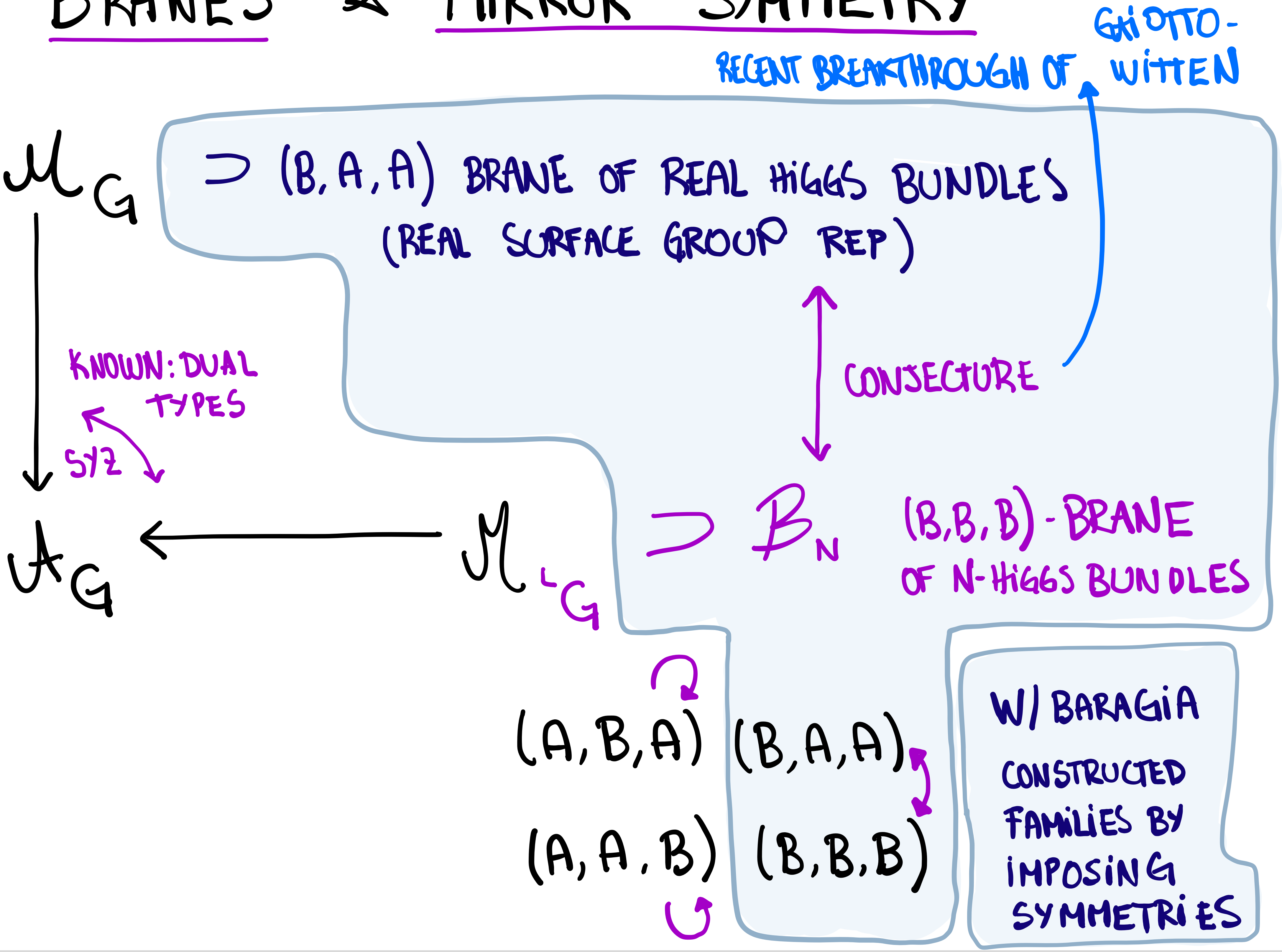
SYZ

HOMOLOGICAL MIRROR SYMMETRY
FUKAYA CATEGORY (A-BRANES)
DUAL TO DERIVE CATEGORY OF
COHERENT SHEAVES (B-BRANE)

$$\begin{tikzcd}
(A,B,A) \arrow[loop above] & (B,A,A) \arrow[d, leftrightarrow, bend left] \\
(A,A,B) \arrow[loop below] & (B,B,B)
\end{tikzcd}$$

W/ BARAGIA
CONSTRUCTED
FAMILIES BY
IMPOSING
SYMMETRIES

BRANES & MIRROR SYMMETRY



$$\begin{tikzcd}
\mathcal{M}_G \arrow[d] & \\
\mathcal{A}_G & \mathcal{M}_{^LG} \arrow[l]
\end{tikzcd}$$

RECENT BREAKTHROUGH OF GAIOTTO-WITTEN

$\mathcal{M}_G \supset$ (B,A,A) BRANE OF REAL HIGGS BUNDLES (REAL SURFACE GROUP REP)

CONJECTURE

KNOWN: DUAL TYPES

SYZ

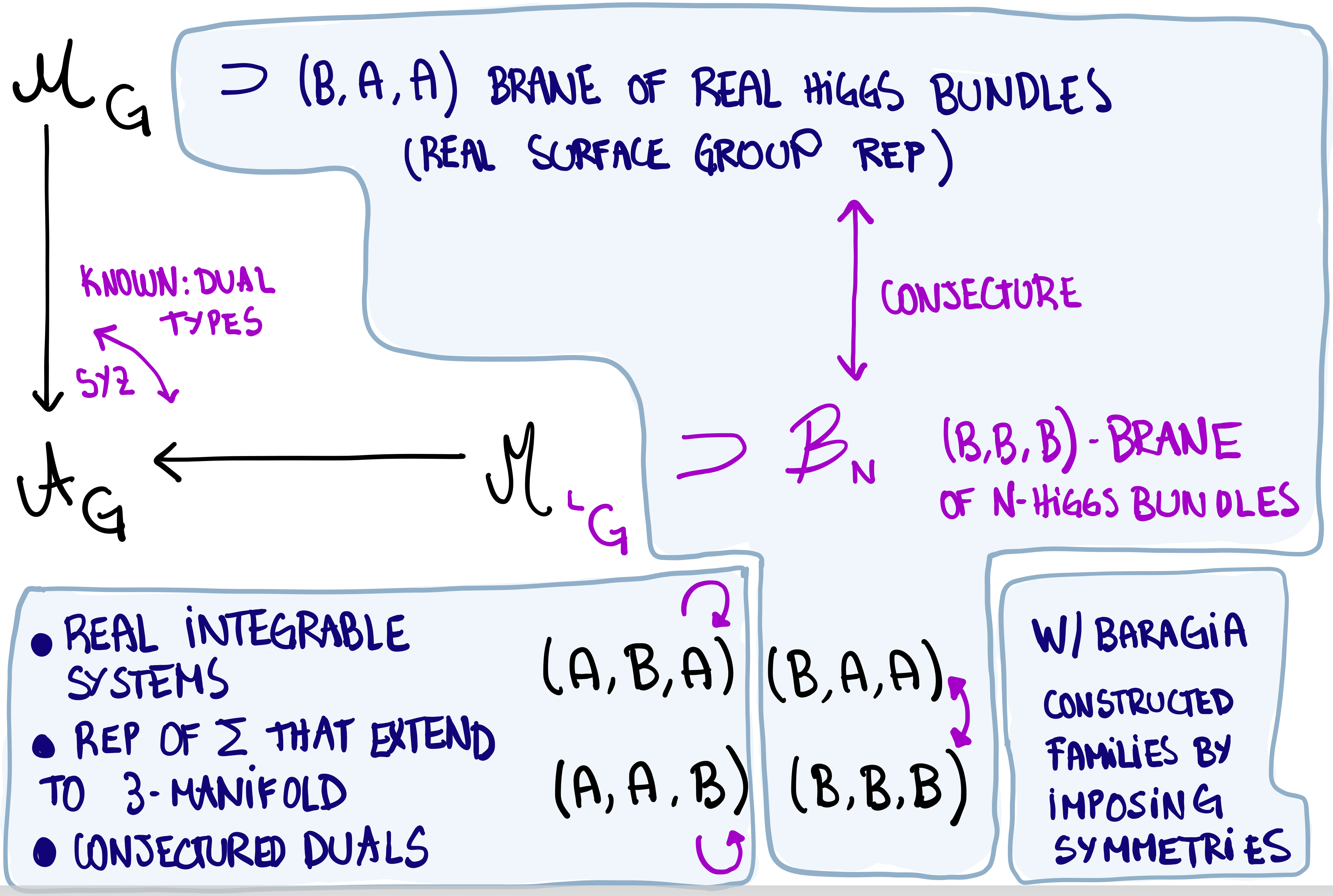
$\mathcal{M}_{^LG} \supset \mathcal{B}_N$ (B,B,B)-BRANE OF N-HIGGS BUNDLES

(A,B,A) (B,A,A)

(A,A,B) (B,B,B)

W/ BARAGIA CONSTRUCTED FAMILIES BY IMPOSING SYMMETRIES

BRANES & MIRROR SYMMETRY



$$\begin{tikzcd}
\mathcal{M}_G \arrow[d] & \\
\mathcal{A}_G & \mathcal{M}_{^LG} \arrow[l]
\end{tikzcd}$$

$\supset$ (B, A, A) BRANE OF REAL HIGGS BUNDLES (REAL SURFACE GROUP REP)

CONJECTURE

$\supset$ $\mathcal{B}_N$ (B,B,B)-BRANE OF N-HIGGS BUNDLES

KNOWN: DUAL TYPES

SYZ

- REAL INTEGRABLE SYSTEMS
- REP OF Σ THAT EXTEND TO 3-MANIFOLD
- CONJECTURED DUALS

(A,B,A) (B,A,A)

(A,A,B) (B,B,B)

W/ BARAGLIA

CONSTRUCTED FAMILIES BY IMPOSING SYMMETRIES

HIGGS BUNDLES

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

+ STABILITY

(E, Φ) ON A COMPACT RIEMANN SURFACE Σ

- * PARABOLIC STRUCTURES
- * TAME/WILD SINGULARITIES

HIGGS BUNDLES

$\mathcal{M}_G \longrightarrow$ MODULI SPACE OF G-HIGGS BUNDLES

+ STABILITY

(E, Φ) ON A $\underbrace{\text{COMPACT RIEMANN SURFACE } \Sigma}$

* PARABOLIC STRUCTURES
* TAME / WILD SINGULARITIES

* HIGHER DIM. MANIFOLDS
* MARKED POINTS
* LOWER GENUS

HIGGS BUNDLES

$\mathcal{M}_G \rightarrow$ MODULI SPACE OF G-HIGGS BUNDLES

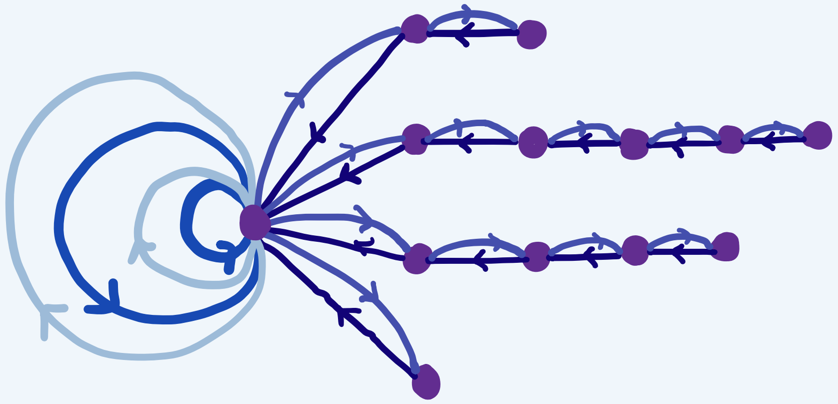
+ STABILITY

(E, Φ) ON A $\underbrace{\text{COMPACT RIEMANN SURFACE } \Sigma}$

- * PARABOLIC STRUCTURES
- * TAME / WILD SINGULARITES

- * HIGHER DIM. MANIFOLDS
- * MARKED POINTS
- * LOWER GENUS

GENERALIZED HYPERPOLYGONS W/RAYAN



HIGGS BUNDLES

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

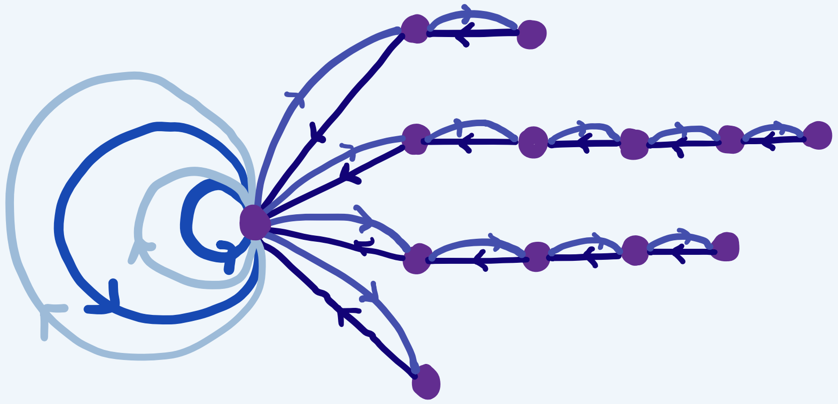
+ STABILITY

(E, Φ) ON A COMPACT RIEMANN SURFACE Σ

- ✱ PARABOLIC STRUCTURES
- ✱ TAME/WILD SINGULARITIES

- ✱ HIGHER DIM. MANIFOLDS
- ✱ MARKED POINTS
- ✱ LOWER GENUS

GENERALIZED HYPERPOLYGONS W/RAYAN



IN CORRESPONDENCE WITH MEROMORPHIC HIGGS BUNDLES

HIGGS BUNDLES

$\mathcal{M}_G$ → MODULI SPACE OF G-HIGGS BUNDLES

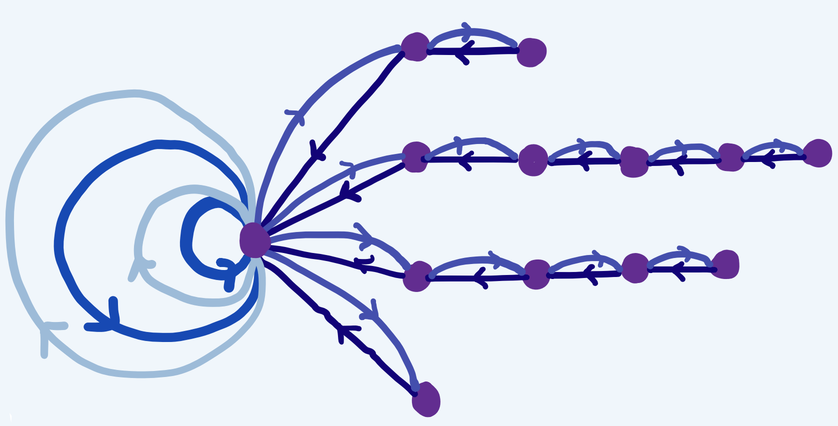
+ STABILITY

(E, Φ) ON A $\underbrace{\text{COMPACT RIEMANN SURFACE } \Sigma}$

- * PARABOLIC STRUCTURES
- * TAME / WILD SINGULARITES

- * HIGHER DIM. MANIFOLDS
- * MARKED POINTS
- * LOWER GENUS

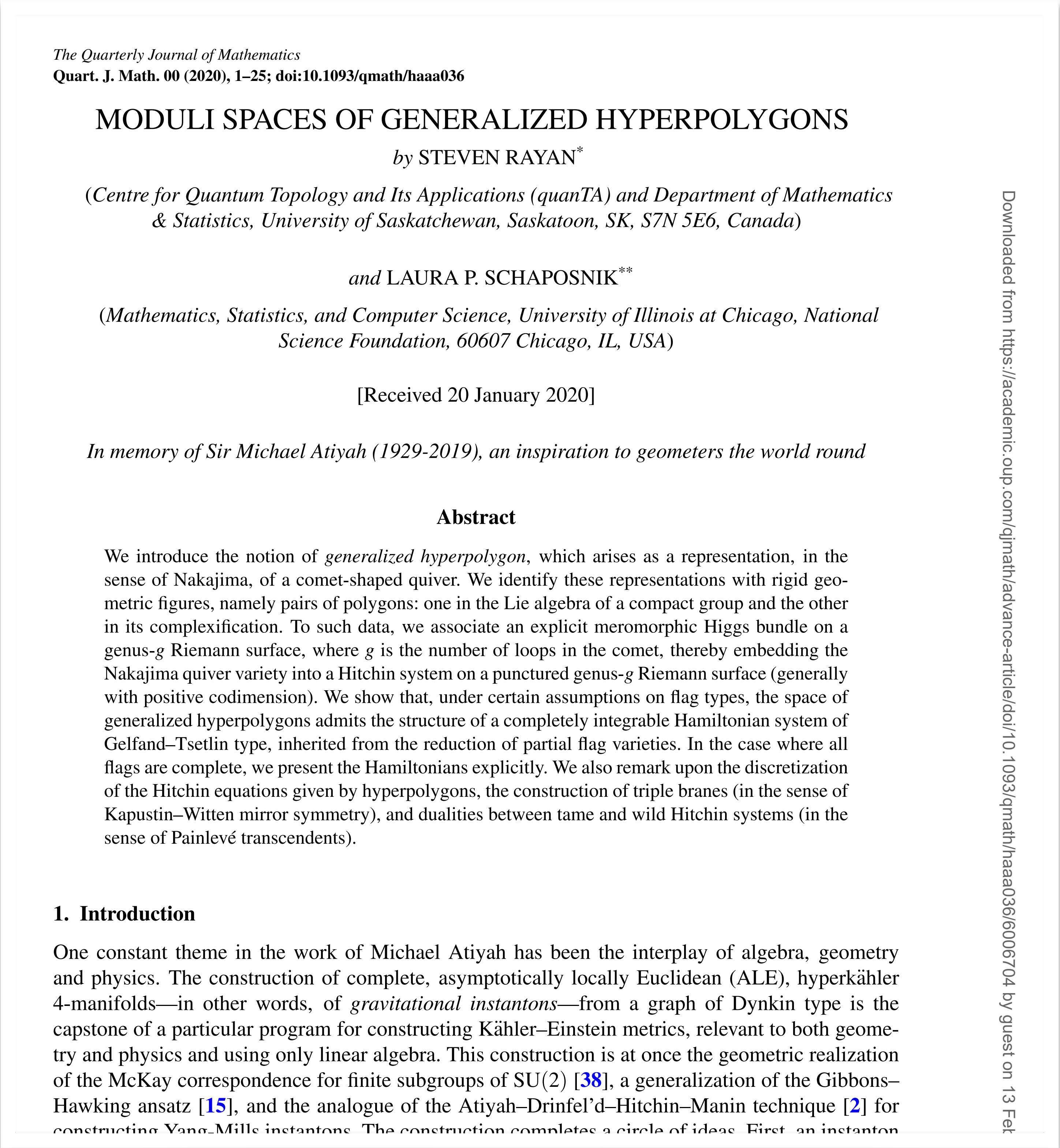
GENERALIZED HYPERPOLYGONS W/RAYAN



IN CORRESPONDENCE WITH MEROMORPHIC HIGGS BUNDLES

ADMITS AN INTEGRABLE SYSTEM AS SUB-INTEGRABLE SYSTEM WITHIN PARABOLIC HIGGS BUNDLES VIA ONLY REPRESENTATION-THEORETIC DATA.

More on generalized Hyperpolygons



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MODULI SPACES OF GENERALIZED HYPERPOLYGONS

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In memory of Sir Michael Atiyah (1929-2019), an inspiration to geometers the world round

Abstract

We introduce the notion of *generalized hyperpolygon*, which arises as a representation, in the sense of Nakajima, of a comet-shaped quiver. We identify these representations with rigid geometric figures, namely pairs of polygons: one in the Lie algebra of a compact group and the other in its complexification. To such data, we associate an explicit meromorphic Higgs bundle on a genus-g Riemann surface, where g is the number of loops in the comet, thereby embedding the Nakajima quiver variety into a Hitchin system on a punctured genus-g Riemann surface (generally with positive codimension). We show that, under certain assumptions on flag types, the space of generalized hyperpolygons admits the structure of a completely integrable Hamiltonian system of Gelfand–Tsetlin type, inherited from the reduction of partial flag varieties. In the case where all flags are complete, we present the Hamiltonians explicitly. We also remark upon the discretization of the Hitchin equations given by hyperpolygons, the construction of triple branes (in the sense of Kapustin–Witten mirror symmetry), and dualities between tame and wild Hitchin systems (in the sense of Painlevé transcendents).

1. Introduction

One constant theme in the work of Michael Atiyah has been the interplay of algebra, geometry and physics. The construction of complete, asymptotically locally Euclidean (ALE), hyperkähler 4-manifolds—in other words, of *gravitational instantons*—from a graph of Dynkin type is the capstone of a particular program for constructing Kähler–Einstein metrics, relevant to both geometry and physics and using only linear algebra. This construction is at once the geometric realization of the McKay correspondence for finite subgroups of SU(2) [38], a generalization of the Gibbons–Hawking ansatz [15], and the analogue of the Atiyah–Drinfel'd–Hitchin–Manin technique [2] for constructing Yang–Mills instantons. The construction completes a circle of ideas. First, an instanton

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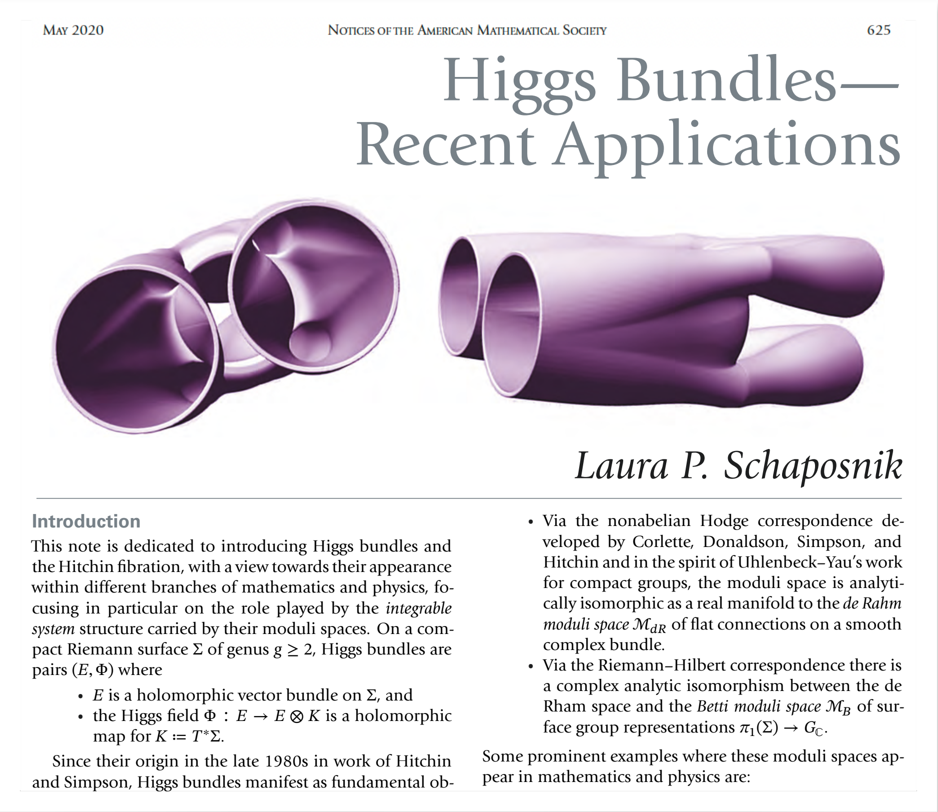


Higgs Bundles—Recent Applications

Laura P. Schaposnik

Introduction

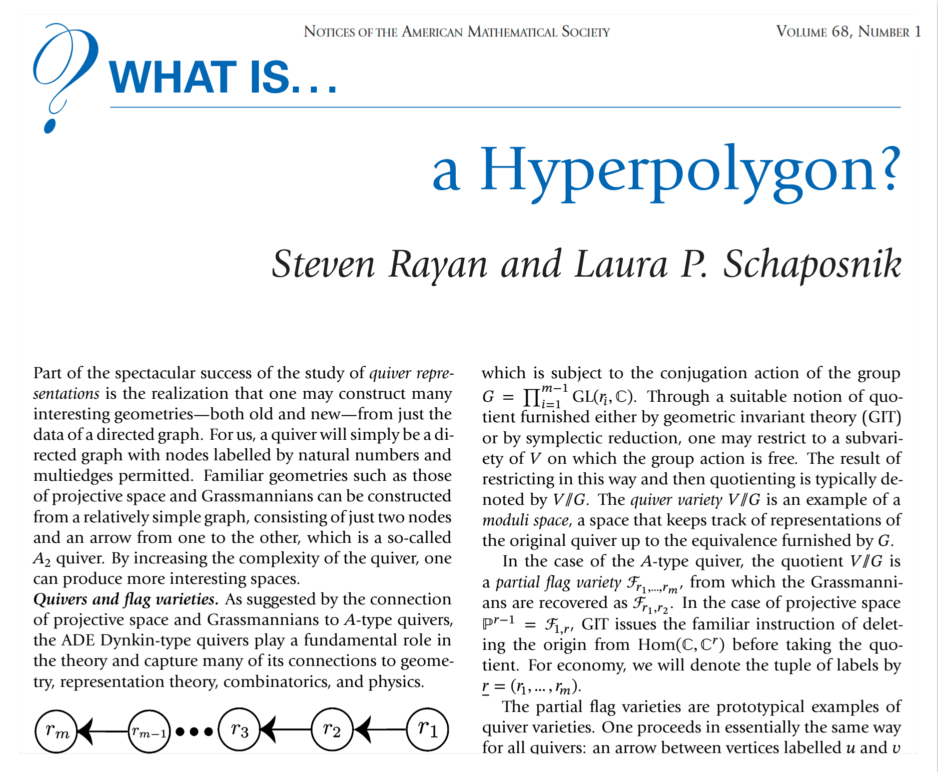
This note is dedicated to introducing Higgs bundles and the Hitchin fibration, with a view towards their appearance within different branches of mathematics and physics, focusing in particular on the role played by the *integrable system* structure carried by their moduli spaces. On a compact Riemann surface Σ of genus $g \geq 2$, Higgs bundles are pairs (E, Φ) where

- E is a holomorphic vector bundle on Σ, and
- the Higgs field $\Phi : E \to E \otimes K$ is a holomorphic map for $K := T^*\Sigma$.

Since their origin in the late 1980s in work of Hitchin and Simpson, Higgs bundles manifest as fundamental ob-

- Via the nonabelian Hodge correspondence developed by Corlette, Donaldson, Simpson, and Hitchin and in the spirit of Uhlenbeck–Yau's work for compact groups, the moduli space is analytically isomorphic as a real manifold to the *de Rahm moduli space* $\mathcal{M}_{dR}$ of flat connections on a smooth complex bundle.
- Via the Riemann–Hilbert correspondence there is a complex analytic isomorphism between the de Rham space and the *Betti moduli space* $\mathcal{M}_B$ of surface group representations $\pi_1(\Sigma) \to G_{\mathbb{C}}$.

Some prominent examples where these moduli spaces appear in mathematics and physics are:



WHAT IS. . .

a Hyperpolygon?

Steven Rayan and Laura P. Schaposnik

Part of the spectacular success of the study of *quiver representations* is the realization that one may construct many interesting geometries—both old and new—from just the data of a directed graph. For us, a quiver will simply be a directed graph with nodes labelled by natural numbers and multiedges permitted. Familiar geometries such as those of projective space and Grassmannians can be constructed from a relatively simple graph, consisting of just two nodes and an arrow from one to the other, which is a so-called A_2 quiver. By increasing the complexity of the quiver, one can produce more interesting spaces.

Quivers and flag varieties. As suggested by the connection of projective space and Grassmannians to A-type quivers, the ADE Dynkin-type quivers play a fundamental role in the theory and capture many of its connections to geometry, representation theory, combinatorics, and physics.

which is subject to the conjugation action of the group $G = \prod_{i=1}^{m-1} \mathrm{GL}(r_i, \mathbb{C})$. Through a suitable notion of quotient furnished either by geometric invariant theory (GIT) or by symplectic reduction, one may restrict to a subvariety of V on which the group action is free. The result of restricting in this way and then quotienting is typically denoted by $V /\!\!/ G$. The *quiver variety* $V /\!\!/ G$ is an example of a *moduli space*, a space that keeps track of representations of the original quiver up to the equivalence furnished by G.

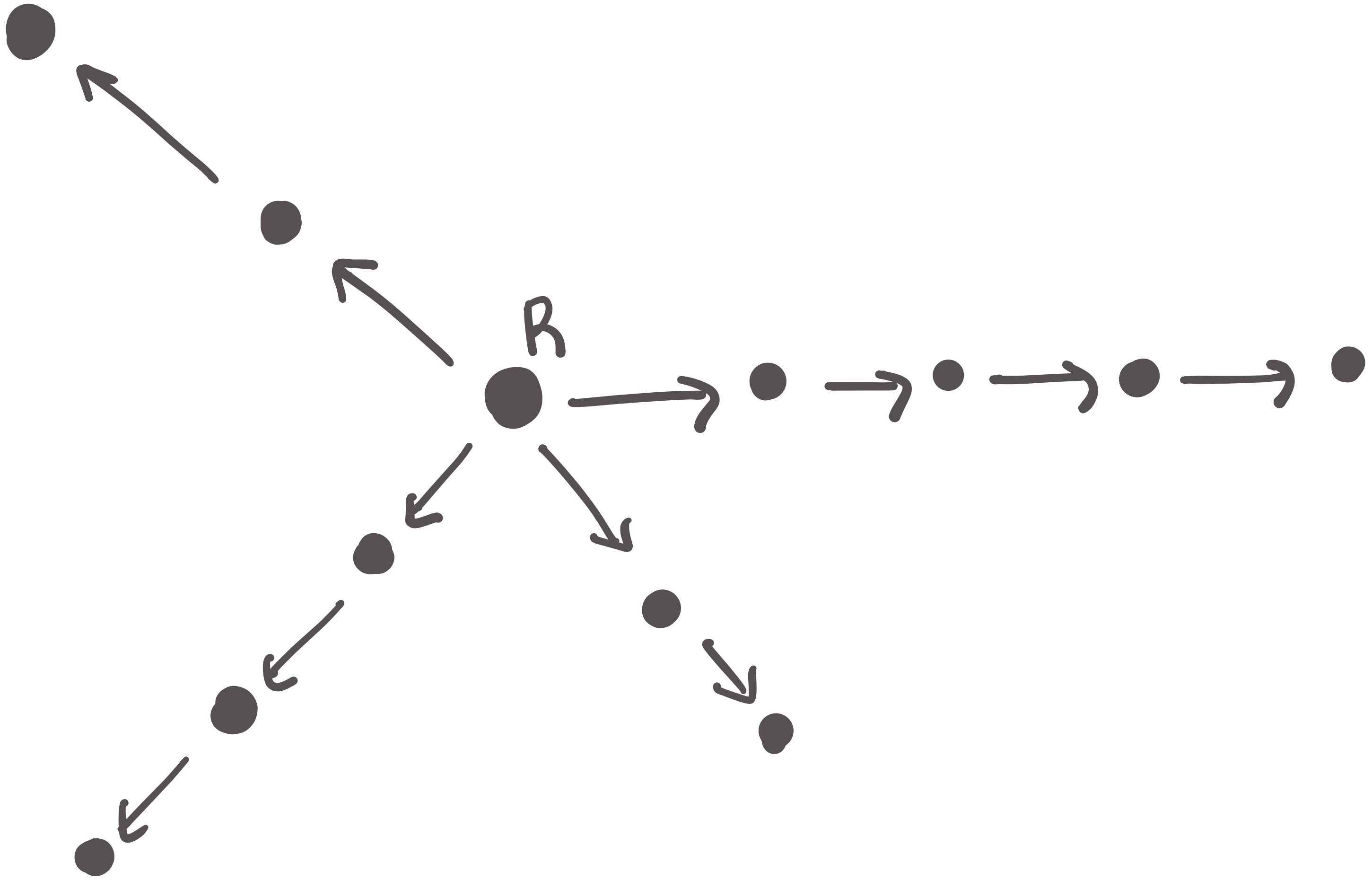
In the case of the A-type quiver, the quotient $V /\!\!/ G$ is a *partial flag variety* $\mathcal{F}_{r_1,\dots,r_m}$, from which the Grassmannians are recovered as $\mathcal{F}_{r_1,r_2}$. In the case of projective space $\mathbb{P}^{r-1} = \mathcal{F}_{1,r}$, GIT issues the familiar instruction of deleting the origin from $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^r)$ before taking the quotient. For economy, we will denote the tuple of labels by $\underline{r} = (r_1, \dots, r_m)$.

The partial flag varieties are prototypical examples of quiver varieties. One proceeds in essentially the same way for all quivers: an arrow between vertices labelled u and v

GENERALIZED HYPERPOLYGONS

GENERALIZED HYPERPOLYGONS

NAKAJIMA QUIVER VARIETY OF STAR SHAPED QUIVERS



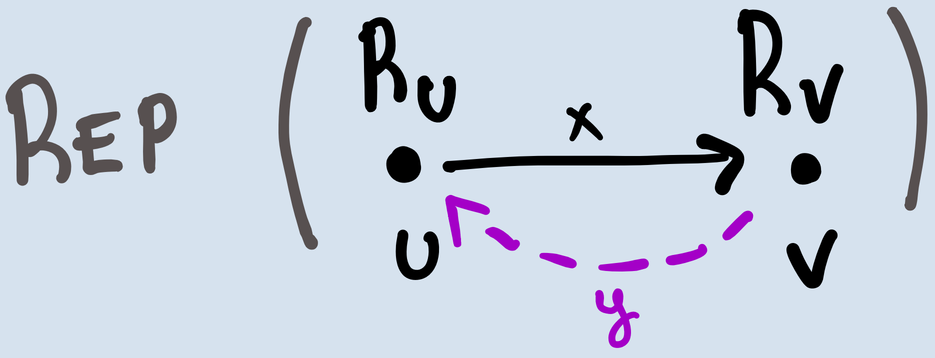
GENERALIZED HYPERPOLYGONS

NAKAJIMA QUIVER VARIETY OF STAR SHAPED QUIVERS

$$\mathrm{REP}\left(\overset{R_u}{\underset{u}{\bullet}} \xrightarrow{x} \overset{R_v}{\underset{v}{\bullet}}\right) = \underbrace{\mathrm{HOM}\left(\mathbb{C}^{R_u}, \mathbb{C}^{R_v}\right)}_{V}$$

GENERALIZED HYPERPOLYGONS

NAKAJIMA QUIVER VARIETY OF STAR SHAPED QUIVERS



NAKAJIMA QUIVER

$$= T^*\underbrace{\mathrm{HOM}(\mathbb{C}^{R_u},\mathbb{C}^{R_v})}_{T^*V}$$

$$\cong \begin{array}{ll} \mathrm{Hom}(\mathbb{C}^{R_u},\mathbb{C}^{R_v}) & \ni x \\ \oplus & \\ \mathrm{Hom}(\mathbb{C}^{R_v},\mathbb{C}^{R_u}) & \ni y \end{array}$$

GENERALIZED HYPERPOLYGONS

SYMPLECTIC REDUCTION

MOMENT MAPS

$$\begin{cases} \mu : T^*\mathrm{REP}\,\Theta_1 \longrightarrow \bigoplus_v \mathfrak{u}(R_v)^* \\ \gamma : T^*\mathrm{REP}\,\Theta_1 \longrightarrow \bigoplus_v \mathfrak{gl}(R_v)^* \end{cases}$$

GENERALIZED HYPERPOLYGONS

SYMPLECTIC REDUCTION

$$\text{MOMENT MAPS}\begin{cases} \mu : T^*\mathrm{REP}\,\Theta \longrightarrow \bigoplus_v \mathfrak{u}(R_v)^* \\[2em] \gamma : T^*\mathrm{REP}\,\Theta \longrightarrow \bigoplus_v \mathfrak{gl}(R_v)^* \end{cases}$$

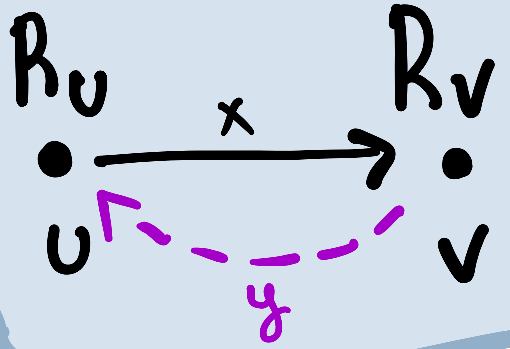
$$G = \Big(\underset{\substack{\uparrow \\ \text{NON} \\ \text{CENTRAL NODES}}}{\prod} U(R_v) \times SU\underset{\substack{\uparrow \\ \text{CENTRAL NODE}}}{(R)}\Big)/\pm 1 \curvearrowright \mathrm{REP}\,\Theta$$

GENERALIZED HYPERPOLYGONS

SYMPLECTIC REDUCTION

MOMENT MAPS

$$\begin{cases} \mu : T^*\mathrm{REP}\,Q \longrightarrow \bigoplus_v \mathfrak{u}(R_v)^* \\ \gamma : T^*\mathrm{REP}\,Q \longrightarrow \bigoplus_v \mathfrak{gl}(R_v)^* \end{cases}$$



$$\mu_u(x,y) = \underbrace{xx^* - yy^*}_{\mathfrak{u}(R_u)^*}$$

$$\gamma_u(x,y) = xy \; \} \; \mathfrak{gl}(R_u)^*$$

GENERALIZED HYPERPOLYGONS

SYMPLECTIC REDUCTION - NAKAJIMA QUIVER VARIETY

$$T^*\mathrm{Rep}(Q) /\!\!/\!\!/_{\alpha} G \qquad G = \prod_v GL(R_v, \mathbb{C})$$

GENERALIZED HYPERPOLYGONS

SYMPLECTIC REDUCTION - NAKAJIMA QUIVER VARIETY

$$T^*\mathrm{REP}(Q) /\!/\!/_{\alpha} G \qquad G = \prod_v GL(R_v, \mathbb{C})$$

LEVEL SETS

$$\mu^{-1}(\alpha) \cap \gamma^{-1}(0) / G$$

$$G = \prod_N U(R_v)/\pm 1$$

$$\alpha \in Z(g^*)$$

HITCHIN· KARHEDE, LINDSON, ROEK

QUAT-COMMUTING CX STRUCTURES

RIEMANNIAN METRIC

GENERALIZED HYPERPOLYGONS

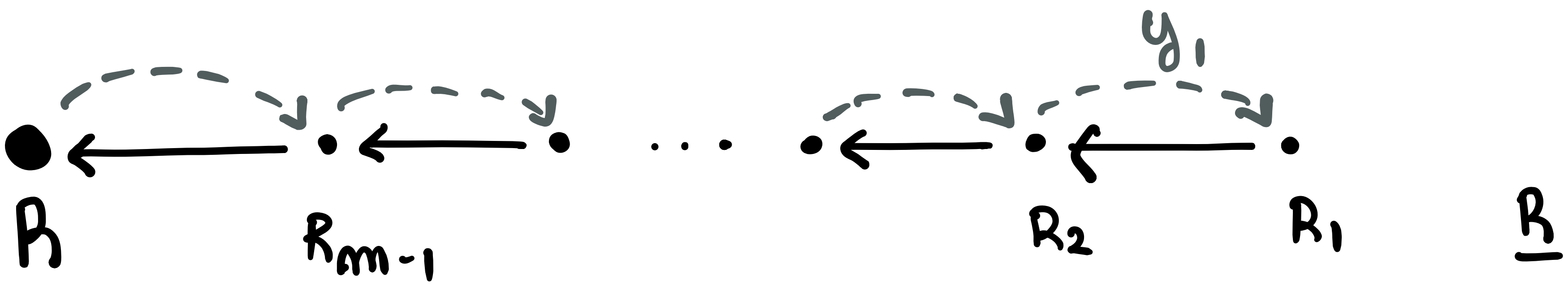
W/ S. RAYAN

COMET SHAPED QUIVERS

GENERALIZED HYPERPOLYGONS

w/ S. RAYAN

COMET SHAPED QUIVERS



PARTIAL FLAG VARIETY

NAKAJIMA QUIVER VARIETY $T^*\mathcal{F}_{\underline{R}}$

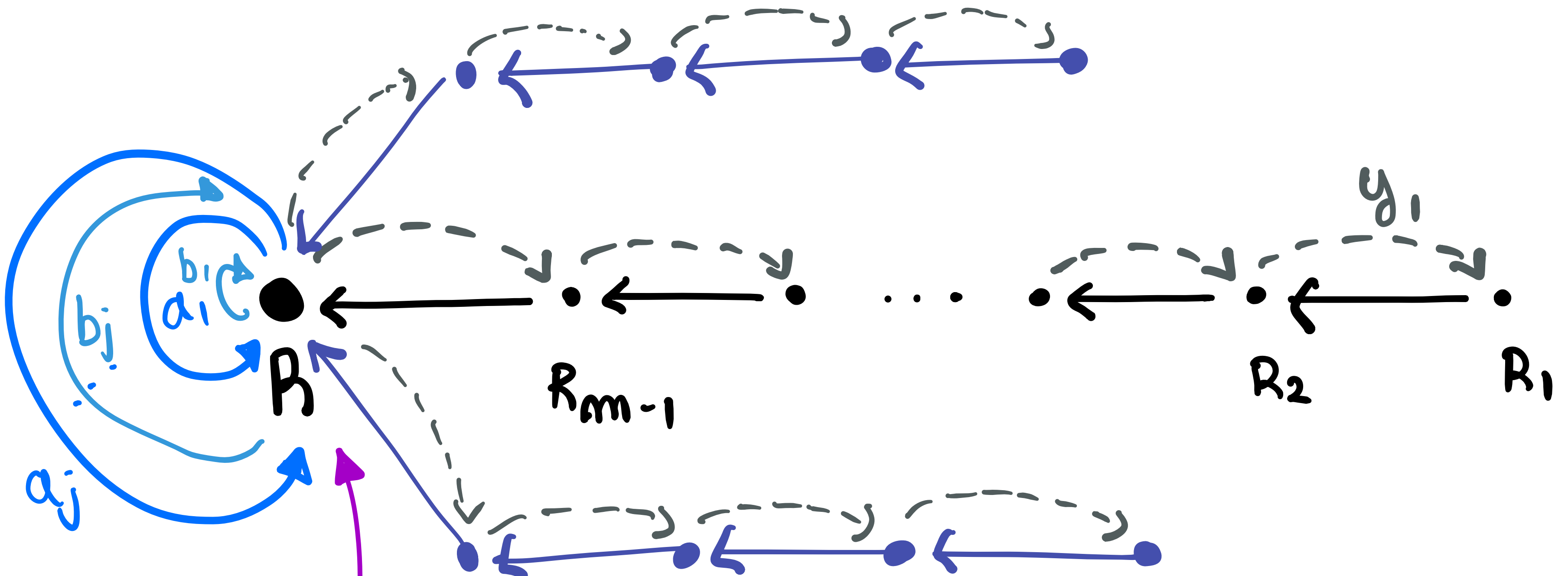
$(\alpha, 0, 0, \ldots)$ $\alpha \in \mathbb{R}$

$\gamma_m(x_{m-1}, y_{m-1}) = x_{m-1}y_{m-1}$

GENERALIZED HYPERPOLYGONS

w/ S. RAYAN

COMET SHAPED QUIVERS



WITH SU(R) ACTION

$\underline{R^1}$ $\alpha_1 \in \mathbb{R}$

$\underline{R^2}$ $\alpha^2 \in \mathbb{R}$

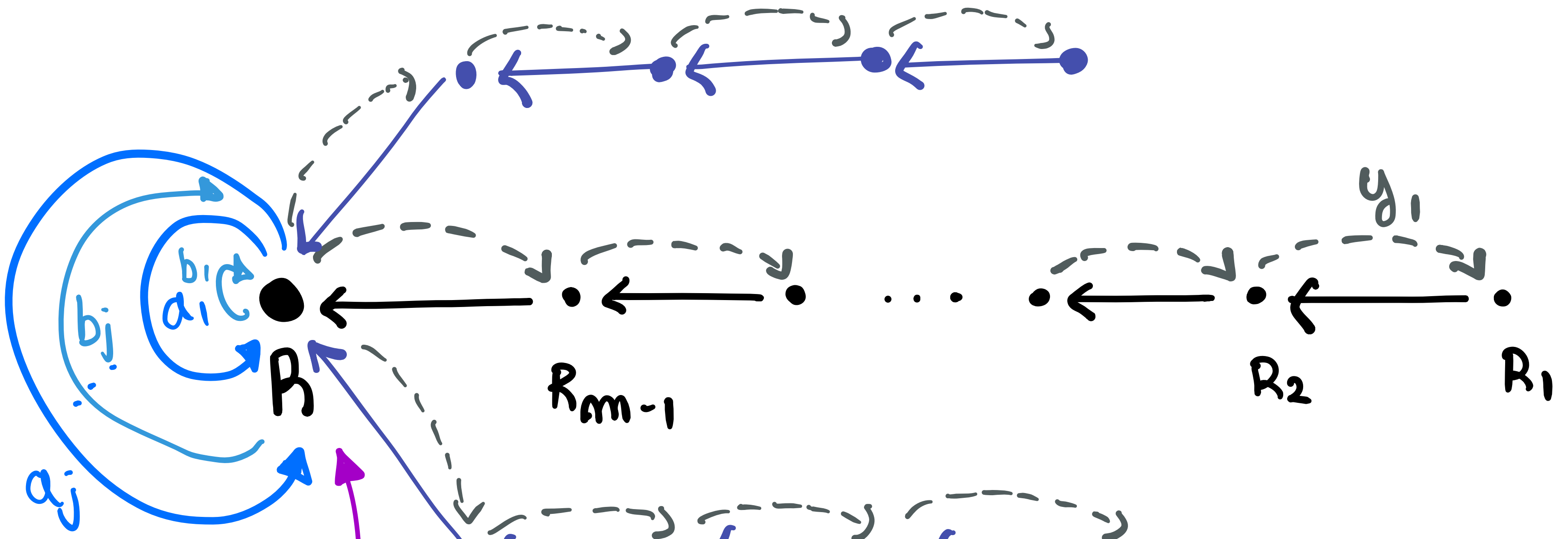
$\vdots$

$\underline{R^m}$ $\alpha^m \in \mathbb{R}$

GENERALIZED HYPERPOLYGONS

↗ W/ S. RAYAN

COMET SHAPED QUIVERS



WITH SU(R) ACTION

$\underline{R}^1$ $\alpha_1 \in \mathbb{R}$

$\underline{R}^2$ $\alpha^2 \in \mathbb{R}$

⋮

$\underline{R}^m$ $\alpha^m \in \mathbb{R}$

$$T^*\mathcal{F}_{\underline{R}^1} \times \dots \times T^*\mathcal{F}_{\underline{R}^m} \times T^*\mathfrak{sl}(R,\mathbb{C})^g /\!/\!/_0 \, SU(R)$$

$$\underline{\alpha} = (\alpha_1, \dots, \alpha^m)$$

GENERALIZED HYPERPOLYGONS

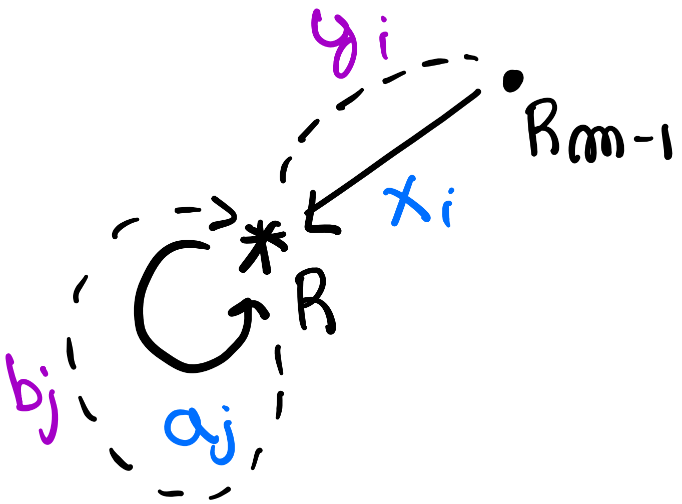
W/ S. RAYAN

COMET SHAPED QUIVERS - THE MOMENT MAPS

GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - THE MOMENT MAPS

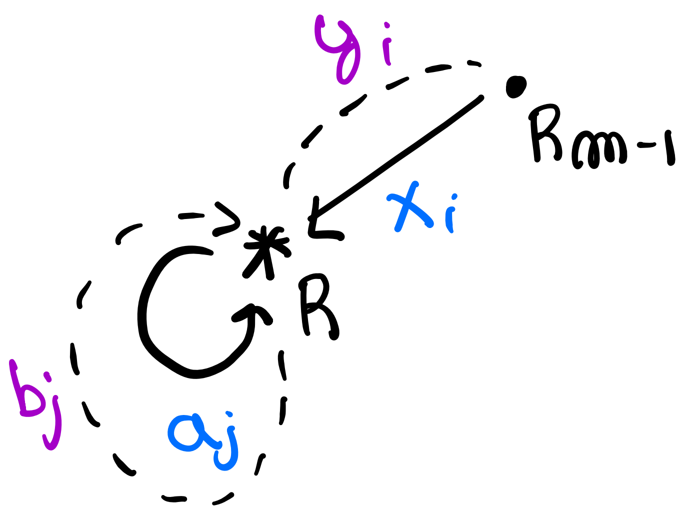


$$\mu_{*}(x,y,a,b) = \sum_{i=1}^{m} (x_i x_i^{*} - y_i^{*} y_i)_0 + \sum_{j=0}^{g} [a_j, a_j^{*}] + [b_j, b_j^{*}]$$

$$\gamma_{*}(x,y,a,b) = \sum_{i=1}^{m} (x_i y_i)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

GENERALIZED HYPERPOLYGONS ↗ W/ S. RAYAN

COMET SHAPED QUIVERS - THE MOMENT MAPS



$$\mu_*(x,y,a,b) = \sum_{i=1}^{m} (x_i x_i^* - y_i^* y_i)_0 + \sum_{j=0}^{g} [a_j, a_j^*] + [b_j, b_j^*]$$

$$\gamma_*(x,y,a,b) = \sum_{i=1}^{m} (x_i y_i)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

MODULI SPACE OF GENERALIZED HYPERPOLYGONS

$$\left.\mathcal{X}^{g}_{\underline{R}^1,\dots,\underline{R}^m}(\alpha) = \frac{\mu_*^{-1}(0) \cap \gamma_*^{-1}(0)}{SU(R)}\right\} \quad 2\left(\sum \dim F_{\underline{R}_i} + (g-1)(R^2-1)\right)$$

w dim

↗ HYPERKHALER VARIETY

GENERALIZED HYPERPOLYGONS

↗ W/ S. RAYAN

COMET SHAPED QUIVERS - HITCHIN SYSTEMS

$$\mu_*(x,y,a,b) = \sum_{i=1}^{m} (x_i x_i^* - y_i^* y_i)_0 + \sum_{j=0}^{g} [a_j, a_j^*] + [b_j, b_j^*]$$

$$\parallel$$

$$0$$

$$\parallel$$

$$\gamma_*(x,y,a,b) = \sum_{i=1}^{m} (x_i y_i)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - HITCHIN SYSTEMS

$$F_A + \phi \wedge \phi^* = 0 \quad \left\{ \left(\sum_{i=1}^{m} (x_i x_i^*)_0 + \sum_{j=0}^{g} [a_j, a_j^*] \right) + \left(\sum_{i=1}^{m} - y_i^* y_i + \sum_{j=0}^{g} [b_j, b_j^*] \right) = 0 \right.$$

$$\bar{\partial}_A \phi \text{ HOLOMORPHIC} \quad \left\{ \sum_{i=1}^{m} (x_i y_i)_0 = - \sum_{j=1}^{g} [a_j, b_j] \right.$$

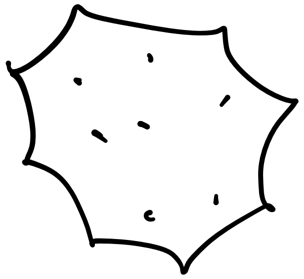
GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - HITCHIN SYSTEMS

$$D=\sum_i z_i$$

4g-gon



$\subset \mathbb{H}$ ($\mathbb{C}$ if $g=0,1$)

Higgs FIELD

$$\phi(z)=\sum_{i=1}^{m}\frac{x_i\, g_i}{z-g_z(z_i)}\,dz$$

QUASI PERIODIC FUNCTIONS

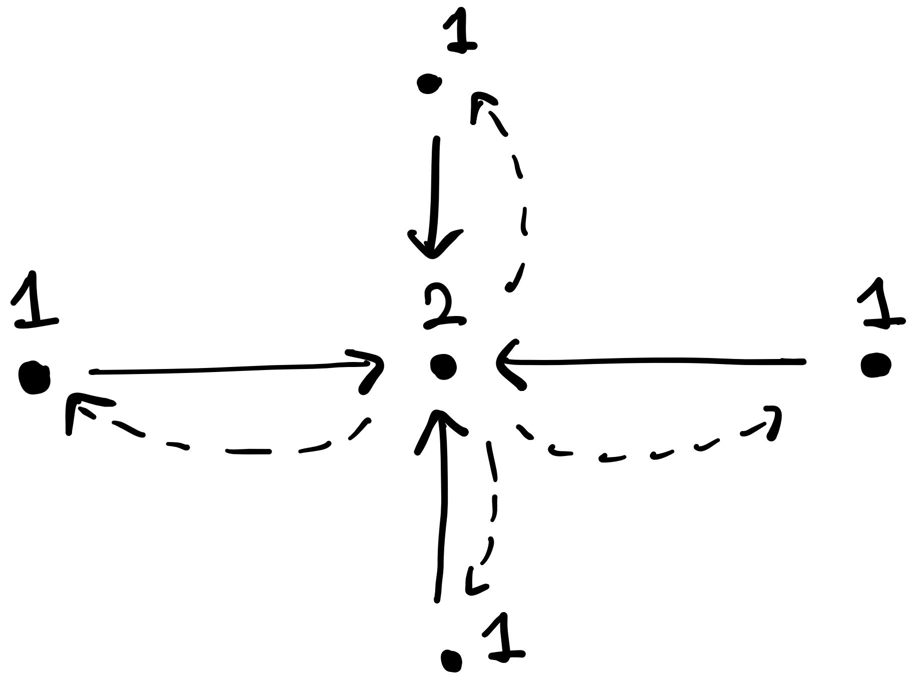
FOR TRIVIAL RANK R BUNDLE ON $X=\mathbb{H}/\Gamma$

GENERALIZED HYPERPOLYGONS

↗ W/ S. RAYAN

COMET SHAPED QUIVERS - HITCHIN SYSTEMS

EXAMPLE (NON GENERALIZED)



AFFINE D_4

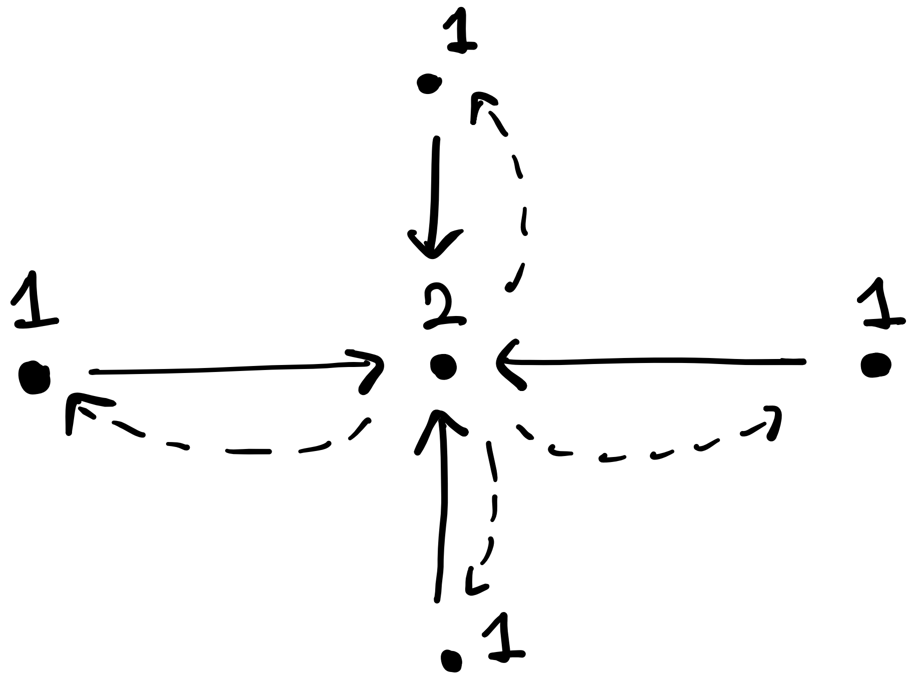
FLAGS $\underline{B^i} = (1,2) = [2]$

GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - HITCHIN SYSTEMS

EXAMPLE (NON GENERALIZED)



AFFINE D_4

FLAGS $\underline{B}^i = (1,2) = [2]$

$\chi_{[2],[2],[2],[2]}(\underline{\alpha})$

- K_3 SURFACE W/ COMPLETE ALE METRIC
- EMBEDS INTO HITCHIN SYST. ON $\mathbb{P}^1 / \{z_1, z_2, z_3, z_4\}$

(PARABOLIC RK 2 W/ 4 TAME SINGULARITIES)

- NOT HYPERKÄHLER EMBEDDING

GENERALIZED HYPERPOLYGONS ↗ W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\begin{tikzcd}
T^*\mathcal{F}_{\underline{\beta}^1} \times T^*\mathcal{F}_{\underline{\beta}^2} \times \cdots \times T^*\mathcal{F}_{\underline{\beta}^m} \times T^*SL(r,\mathbb{C})^g \arrow[d] \\
\mathcal{X}^g_{\underline{\beta}^1,\underline{\beta}^2,\ldots,\underline{\beta}^m}(\underline{\alpha})
\end{tikzcd}$$

GENERALIZED HYPERPOLYGONS

↗ W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\underbrace{T^*\mathcal{F}_{\underline{R}^1} \times T^*\mathcal{F}_{\underline{R}^2} \times \dots \times T^*\mathcal{F}_{\underline{R}^n}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*SL(r,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}}$$

↗ EACH HYPERKAHLER VARIETY

$$\big\downarrow$$

$$X^g_{\underline{R}^1,\underline{R}^2,\dots,\underline{R}^m}(\underline{\alpha})$$

} INHERITS SYMPLECTIC STRUCTURE

GENERALIZED HYPERPOLYGONS ↗ W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

EACH HYPERKAHLER VARIETY

$$\underbrace{T^*\mathcal{F}_{\underline{R}^1} \times T^*\mathcal{F}_{\underline{R}^2} \times \dots \times T^*\mathcal{F}_{\underline{R}^m}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*Sl(R,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}}$$

$$\downarrow$$

$$X^g_{\underline{R}^1,\underline{R}^2,\dots,\underline{R}^m}(\underline{\alpha})$$

} INHERITS SYMPLECTIC STRUCTURE

REMARK: FISHER-RAYAN
$[1,R]$, $m \leq 3$
USING Higgs on
PUNCTURED SPHERE

GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\begin{tikzcd}
\underbrace{T^*\mathcal{F}_{\underline{\beta}^1} \times T^*\mathcal{F}_{\underline{\beta}^2} \times \dots \times T^*\mathcal{F}_{\underline{\beta}^n}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*SL(r,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}} \arrow[d] \\
\mathcal{X}^g_{\underline{\beta}^1,\underline{\beta}^2,\dots,\underline{\beta}^n}(\underline{\alpha})
\end{tikzcd}$$

GENERALIZED HYPERPOLYGONS ↗ W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\begin{tikzcd}
\underbrace{T^*\mathcal{F}_{\underline{\beta}^1} \times T^*\mathcal{F}_{\underline{\beta}^2} \times \dots \times T^*\mathcal{F}_{\underline{\beta}^n}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*SL(r,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}} \arrow[d] \\
X^g_{\underline{\beta}^1,\underline{\beta}^2,\dots,\underline{\beta}^n}(\underline{\alpha})
\end{tikzcd}$$

INVARIANTS

- FROM COMPLETE FLAGS
- FROM MINIMAL FLAGS
- FROM LOOPS

GENERALIZED HYPERPOLYGONS

↗ W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\begin{tikzcd}
\underbrace{T^*F_{\underline{B}^1} \times T^*F_{\underline{B}^2} \times \dots \times T^*F_{\underline{B}^m}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*Sl(n,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}} \arrow[d] \\
X^g_{\underline{B}^1, \underline{B}^2, \dots, \underline{B}^m}(\underline{\alpha})
\end{tikzcd}$$

INVARIANTS

- FROM COMPLETE FLAGS
- FROM MINIMAL FLAGS
- FROM LOOPS

$$\underbrace{\qquad\qquad\qquad\qquad}$$

INVARIANTS $= \frac{1}{2}$ DIM

- FIXED BY SU(R) RED.

GENERALIZED HYPERPOLYGONS

W/ S. RAYAN

COMET SHAPED QUIVERS - INTEGRABLE SYSTEMS

$$\begin{tikzcd} \underbrace{T^*\mathcal{F}_{\underline{B}^1} \times T^*\mathcal{F}_{\underline{B}^2} \times \dots \times T^*\mathcal{F}_{\underline{B}^m}}_{\text{GELFAND-TSEYLIN TYPE}} \times \underbrace{T^*SL(r,\mathbb{C})^g}_{\text{LIE-POISSON TYPE}} \arrow[d] \\ \mathcal{X}^g_{\underline{B}^1,\underline{B}^2,\dots,\underline{B}^m}(\underline{\alpha}) \end{tikzcd}$$

INVARIANTS

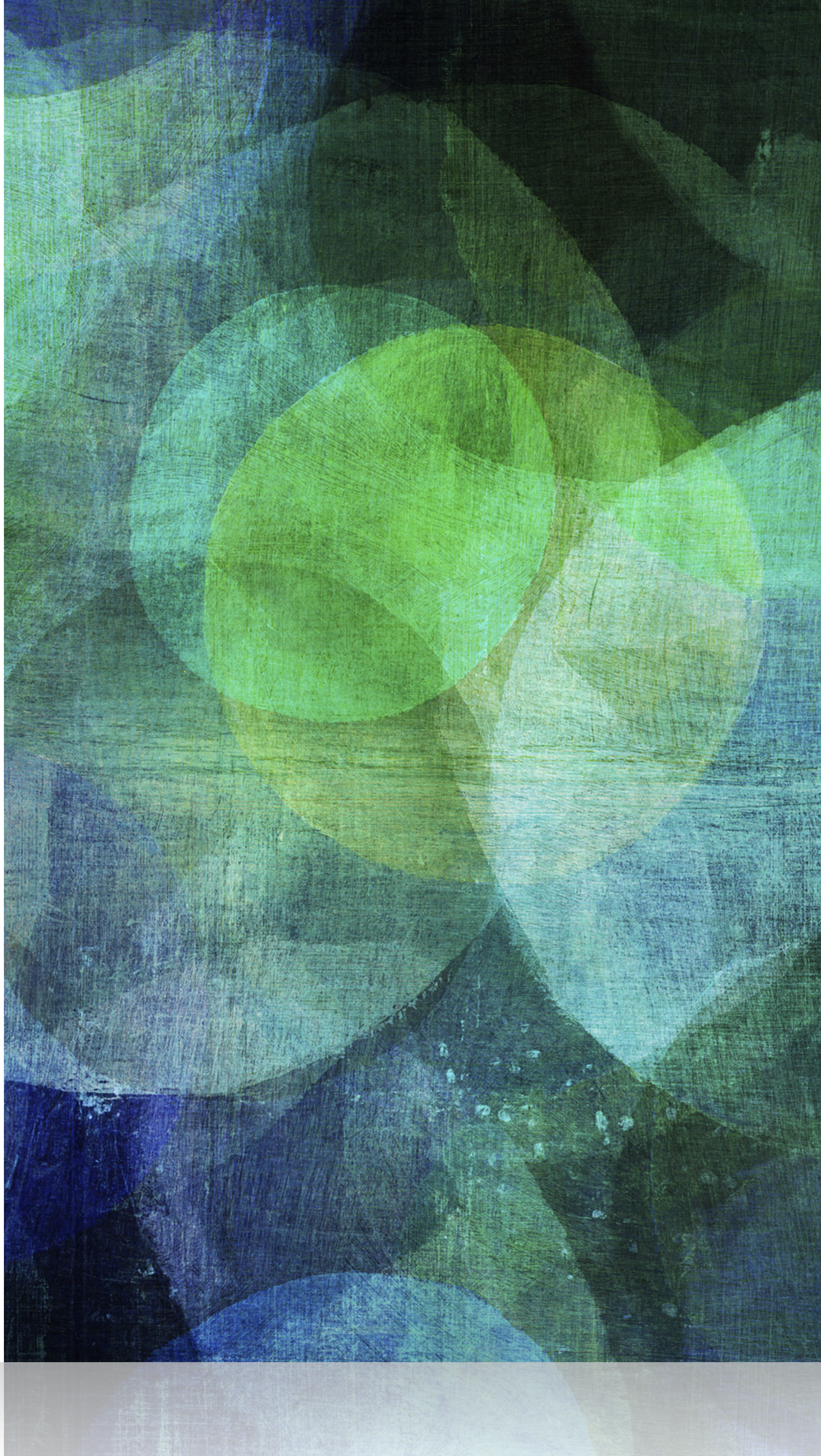
$$\underbrace{\begin{array}{l} \text{- FROM COMPLETE FLAGS} \\ \text{- FROM MINIMAL FLAGS} \\ \text{- FROM LOOPS} \end{array}}$$

\# INVARIANTS - FIXED BY SU(R) RED.

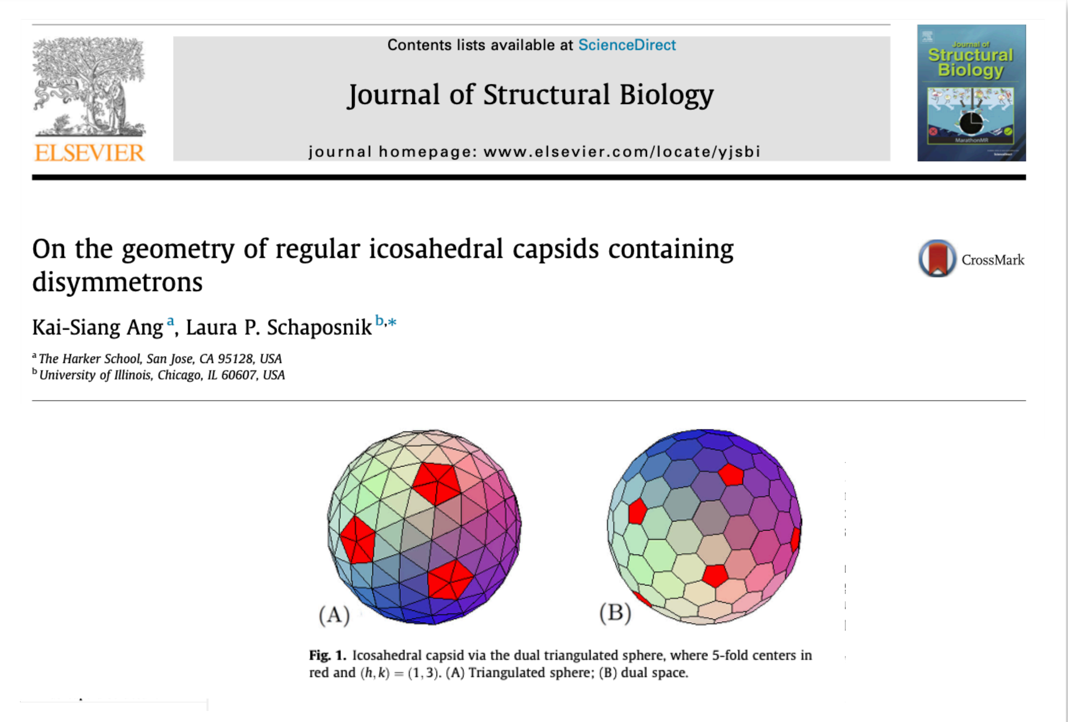
= 1/2 DIM

THEOREM (W/ RAYAN) : FOR COMPLETE OR MINIMAL FLAGS THE SPACE $\mathcal{X}$ IS A COMPLETELY INTEGRABLE SYSTEM OF GELFAND-TSEYLIN TYPE

COME AND TALK TO ME ABOUT...



VIRUSES AND EPIDEMICS



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On the geometry of regular icosahedral capsids containing disymmetrons

Kai-Siang Ang[a], Laura P. Schaposnik[b,*]

[a] The Harker School, San Jose, CA 95128, USA
[b] University of Illinois, Chicago, IL 60607, USA

Fig. 1. Icosahedral capsid via the dual triangulated sphere, where 5-fold centers in red and $(h, k) = (1, 3)$. (A) Triangulated sphere; (B) dual space.

PROCEEDINGS A

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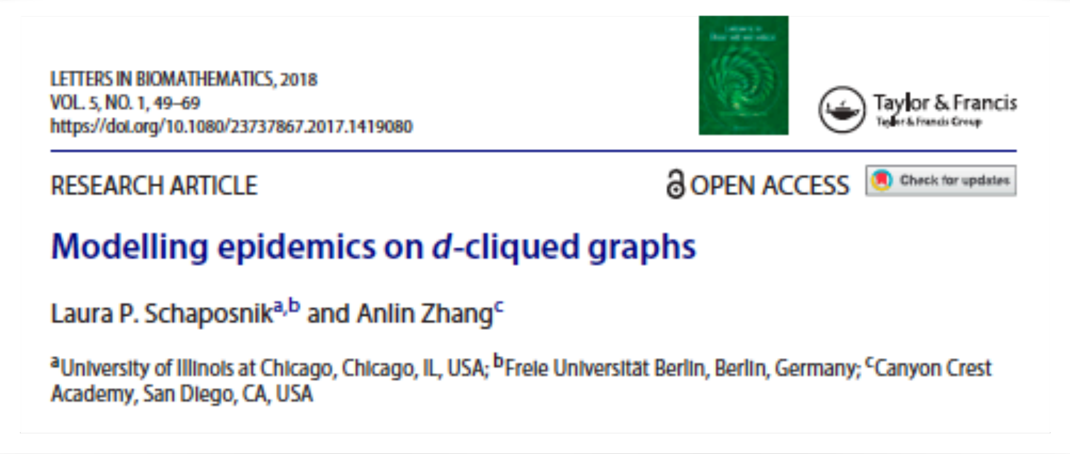
Research

A trust model for spreading gossip in social networks: a multi-type bootstrap percolation model

Rinni Bhansali[1] and Laura P. Schaposnik[2,3]

Cite this article: Bhansali R, Schaposnik LP.



LETTERS IN BIOMATHEMATICS, 2018
VOL. 5, NO. 1, 49–69
https://doi.org/10.1080/23737867.2017.1419080

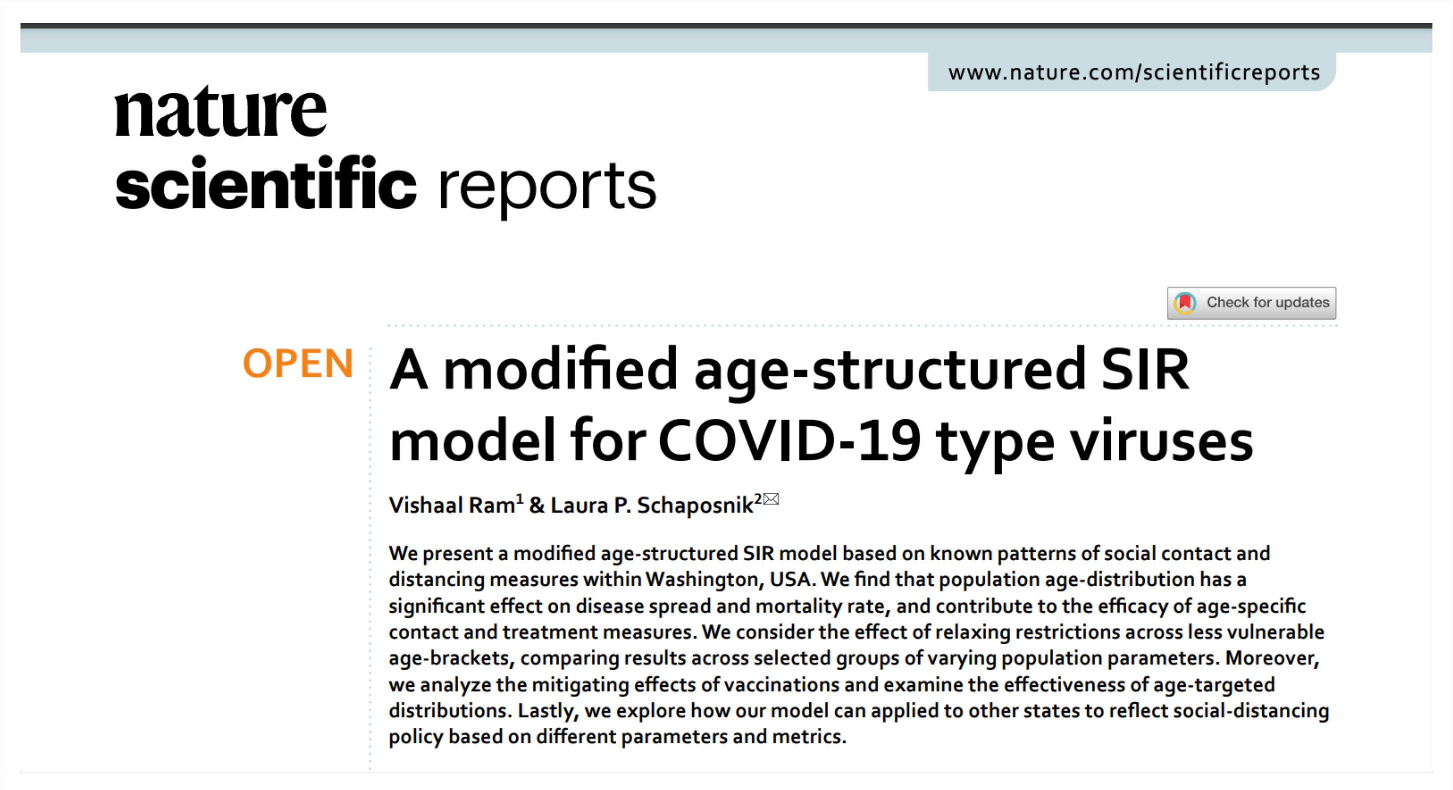
RESEARCH ARTICLE

OPEN ACCESS

Modelling epidemics on *d*-cliqued graphs

Laura P. Schaposnik[a,b] and Anlin Zhang[c]

[a] University of Illinois at Chicago, Chicago, IL, USA; [b] Freie Universität Berlin, Berlin, Germany; [c] Canyon Crest Academy, San Diego, CA, USA



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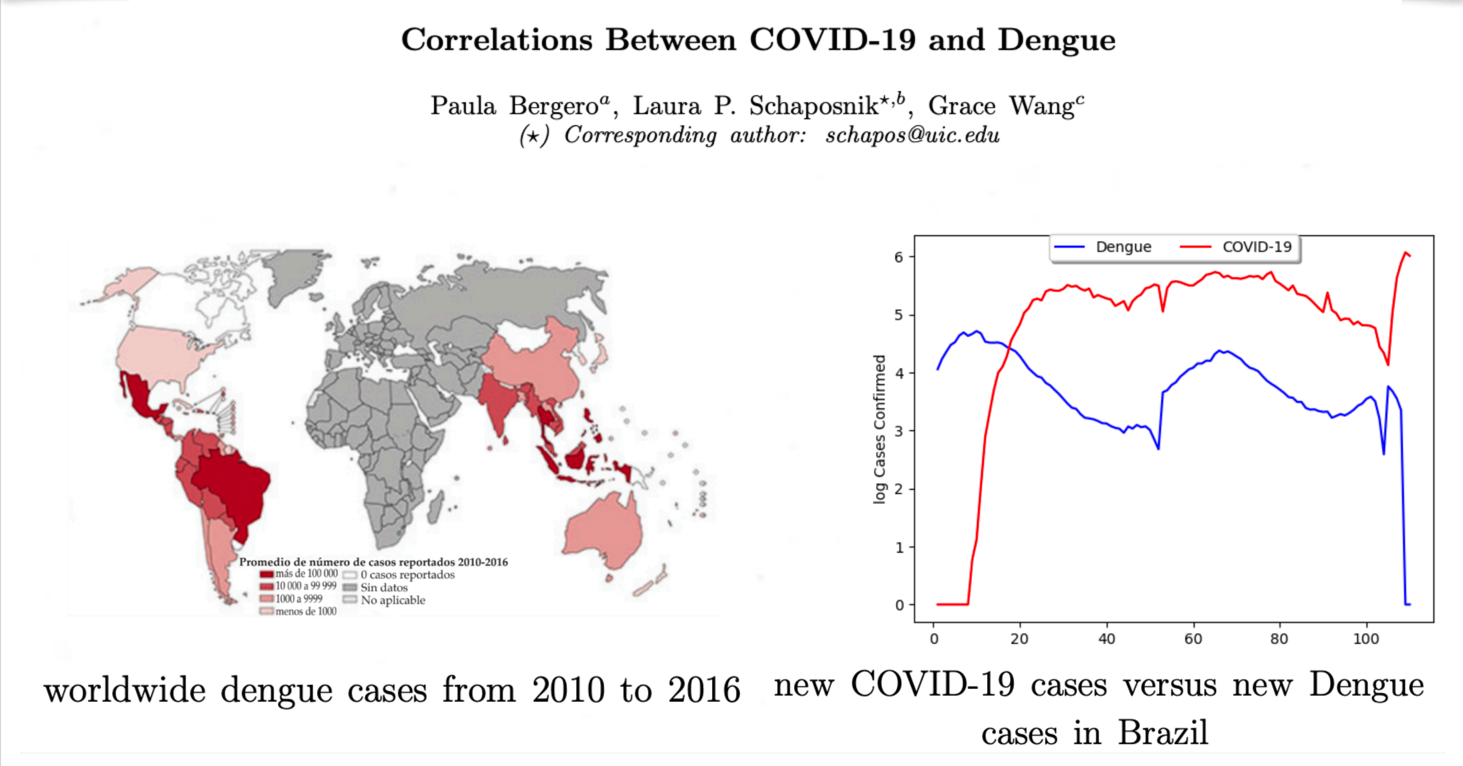
OPEN

A modified age-structured SIR model for COVID-19 type viruses

Vishaal Ram[1] & Laura P. Schaposnik[2]✉

We present a modified age-structured SIR model based on known patterns of social contact and distancing measures within Washington, USA. We find that population age-distribution has a significant effect on disease spread and mortality rate, and contribute to the efficacy of age-specific contact and treatment measures. We consider the effect of relaxing restrictions across less vulnerable age-brackets, comparing results across selected groups of varying population parameters. Moreover, we analyze the mitigating effects of vaccinations and examine the effectiveness of age-targeted distributions. Lastly, we explore how our model can applied to other states to reflect social-distancing policy based on different parameters and metrics.



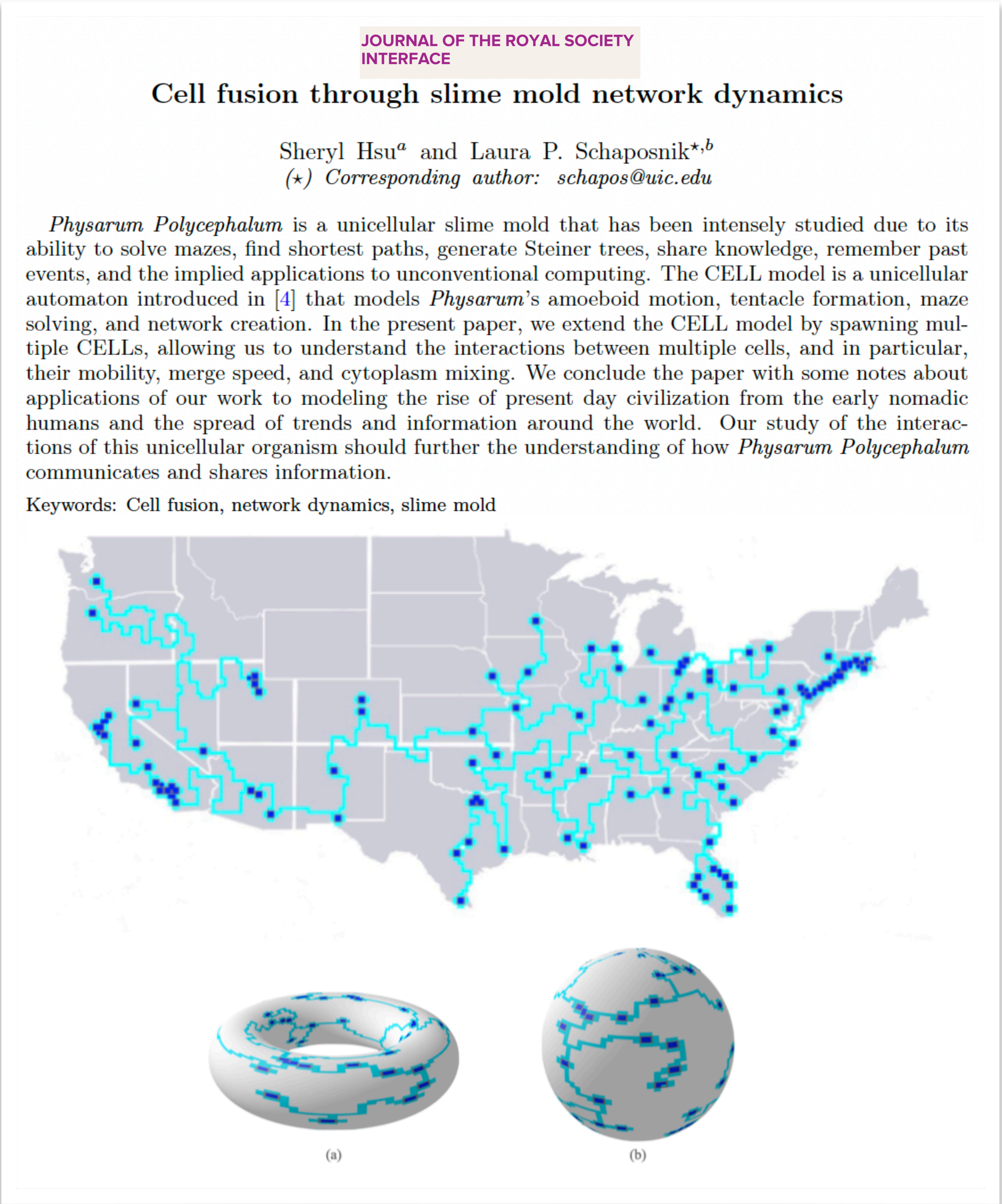
Correlations Between COVID-19 and Dengue

Paula Bergero[a], Laura P. Schaposnik[⋆,b], Grace Wang[c]
(⋆) Corresponding author: schapos@uic.edu

worldwide dengue cases from 2010 to 2016

new COVID-19 cases versus new Dengue cases in Brazil

DYNAMICS IN NATURE



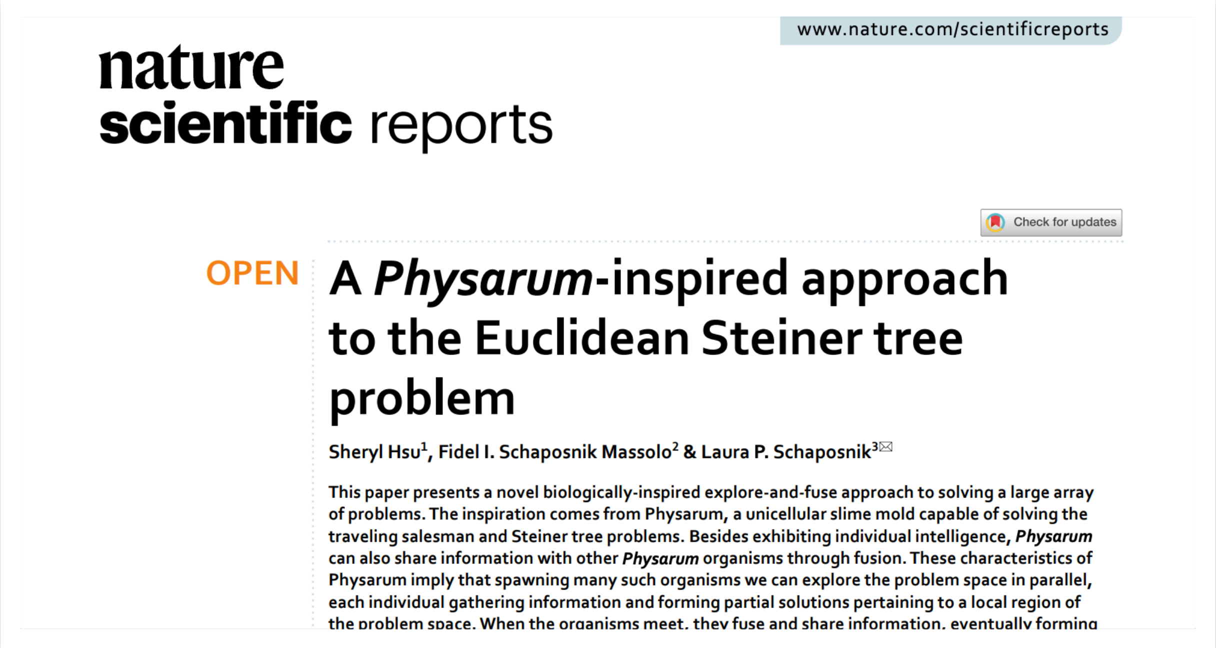
JOURNAL OF THE ROYAL SOCIETY INTERFACE

Cell fusion through slime mold network dynamics

Sheryl Hsu[a] and Laura P. Schaposnik[⋆,b]
(⋆) *Corresponding author: schapos@uic.edu*

Physarum Polycephalum is a unicellular slime mold that has been intensely studied due to its ability to solve mazes, find shortest paths, generate Steiner trees, share knowledge, remember past events, and the implied applications to unconventional computing. The CELL model is a unicellular automaton introduced in [4] that models *Physarum*'s amoeboid motion, tentacle formation, maze solving, and network creation. In the present paper, we extend the CELL model by spawning multiple CELLs, allowing us to understand the interactions between multiple cells, and in particular, their mobility, merge speed, and cytoplasm mixing. We conclude the paper with some notes about applications of our work to modeling the rise of present day civilization from the early nomadic humans and the spread of trends and information around the world. Our study of the interactions of this unicellular organism should further the understanding of how *Physarum Polycephalum* communicates and shares information.

Keywords: Cell fusion, network dynamics, slime mold



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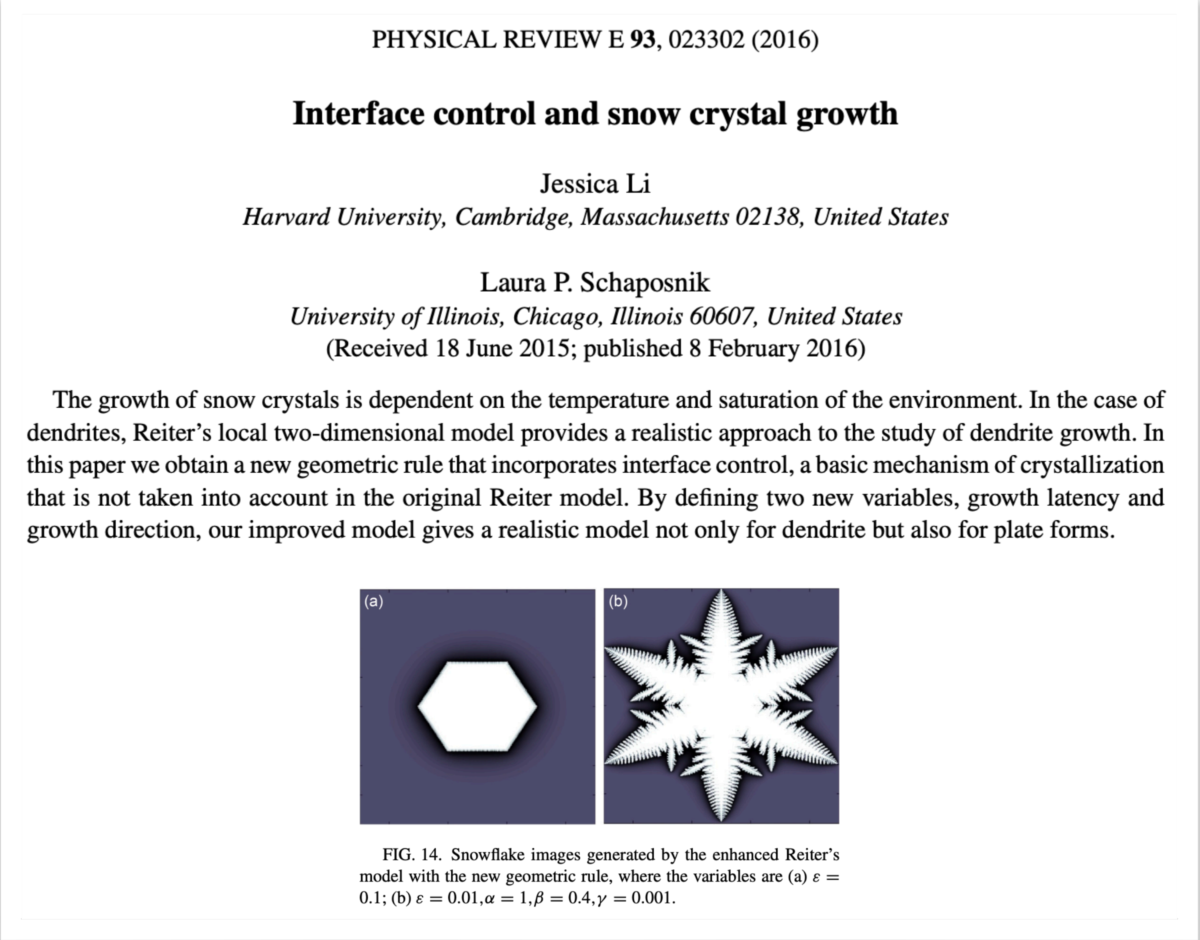
OPEN

A *Physarum*-inspired approach to the Euclidean Steiner tree problem

Sheryl Hsu[1], Fidel I. Schaposnik Massolo[2] & Laura P. Schaposnik[3]✉

This paper presents a novel biologically-inspired explore-and-fuse approach to solving a large array of problems. The inspiration comes from Physarum, a unicellular slime mold capable of solving the traveling salesman and Steiner tree problems. Besides exhibiting individual intelligence, *Physarum* can also share information with other *Physarum* organisms through fusion. These characteristics of Physarum imply that spawning many such organisms we can explore the problem space in parallel, each individual gathering information and forming partial solutions pertaining to a local region of the problem space. When the organisms meet, they fuse and share information, eventually forming



PHYSICAL REVIEW E **93**, 023302 (2016)

Interface control and snow crystal growth

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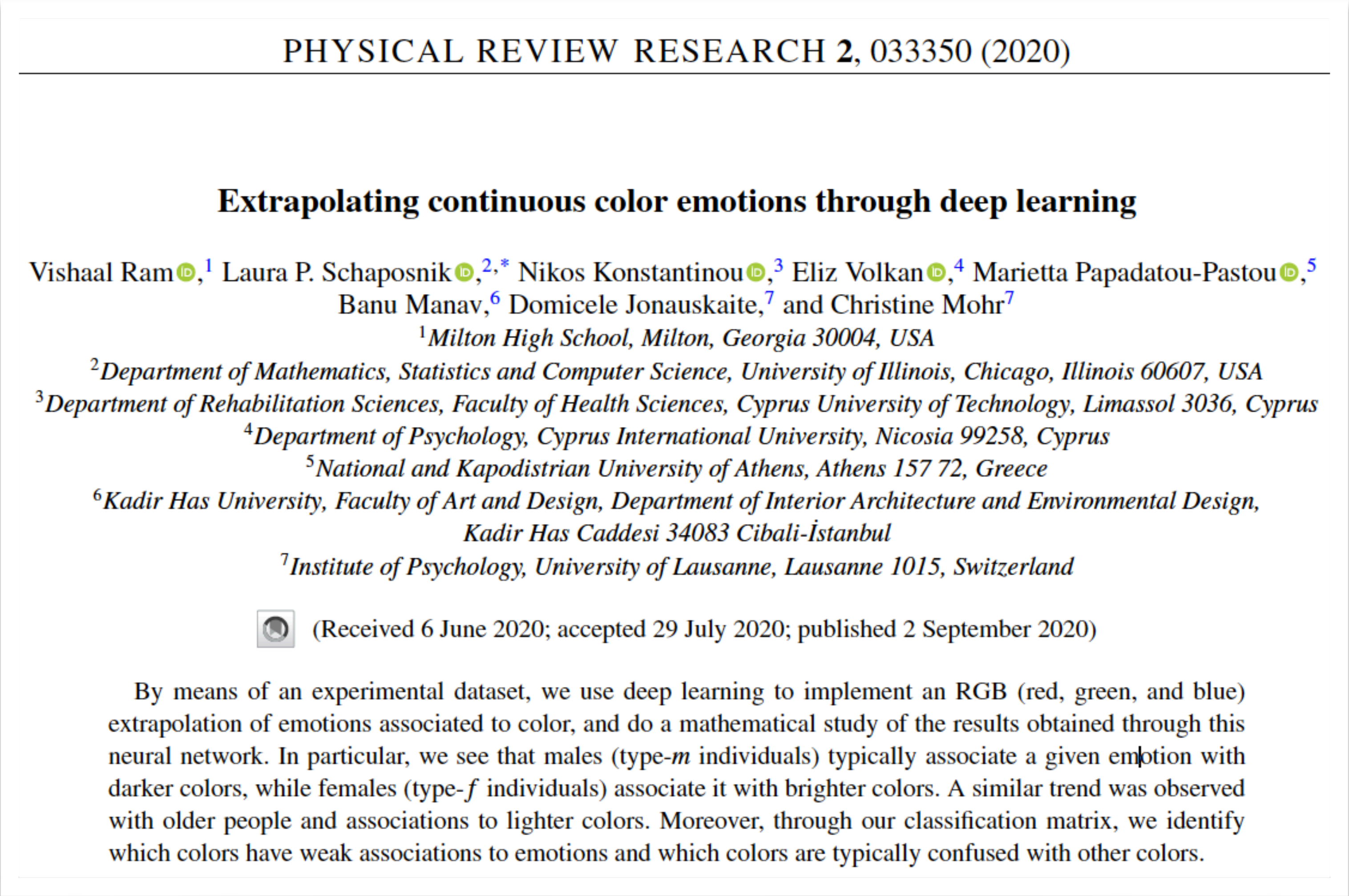
Harvard University, Cambridge, Massachusetts 02138, United States

Laura P. Schaposnik
University of Illinois, Chicago, Illinois 60607, United States
(Received 18 June 2015; published 8 February 2016)

The growth of snow crystals is dependent on the temperature and saturation of the environment. In the case of dendrites, Reiter's local two-dimensional model provides a realistic approach to the study of dendrite growth. In this paper we obtain a new geometric rule that incorporates interface control, a basic mechanism of crystallization that is not taken into account in the original Reiter model. By defining two new variables, growth latency and growth direction, our improved model gives a realistic model not only for dendrite but also for plate forms.

FIG. 14. Snowflake images generated by the enhanced Reiter's model with the new geometric rule, where the variables are (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.01, \alpha = 1, \beta = 0.4, \gamma = 0.001$.

BEHAVIORAL SCIENCE



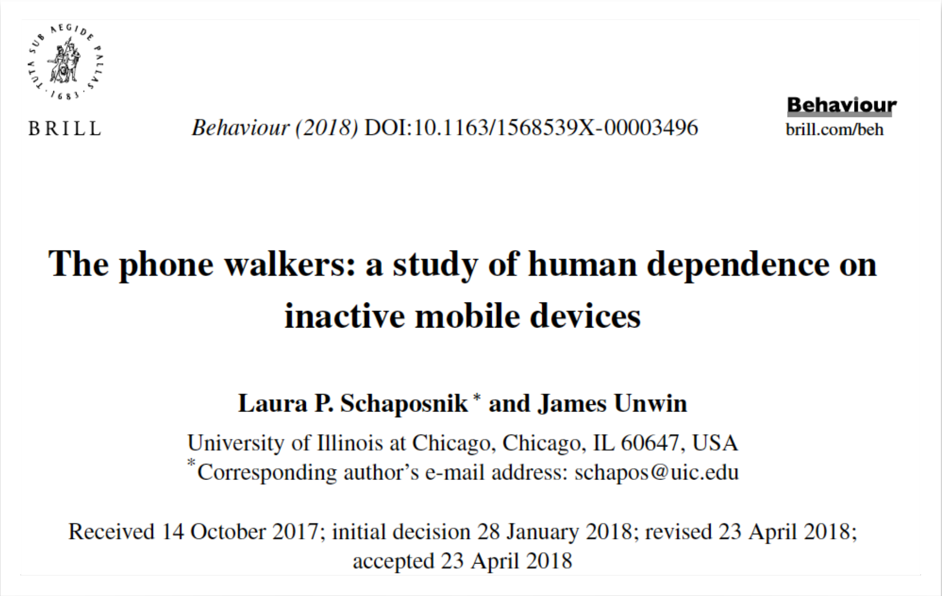
PHYSICAL REVIEW RESEARCH **2**, 033350 (2020)

Extrapolating continuous color emotions through deep learning

Vishaal Ram[1], Laura P. Schaposnik[2,*] Nikos Konstantinou[3] Eliz Volkan[4] Marietta Papadatou-Pastou[5]
Banu Manav[6] Domicele Jonauskaite[7] and Christine Mohr[7]
[1]*Milton High School, Milton, Georgia 30004, USA*
[2]*Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, Illinois 60607, USA*
[3]*Department of Rehabilitation Sciences, Faculty of Health Sciences, Cyprus University of Technology, Limassol 3036, Cyprus*
[4]*Department of Psychology, Cyprus International University, Nicosia 99258, Cyprus*
[5]*National and Kapodistrian University of Athens, Athens 157 72, Greece*
[6]*Kadir Has University, Faculty of Art and Design, Department of Interior Architecture and Environmental Design, Kadir Has Caddesi 34083 Cibali-İstanbul*
[7]*Institute of Psychology, University of Lausanne, Lausanne 1015, Switzerland*

(Received 6 June 2020; accepted 29 July 2020; published 2 September 2020)

By means of an experimental dataset, we use deep learning to implement an RGB (red, green, and blue) extrapolation of emotions associated to color, and do a mathematical study of the results obtained through this neural network. In particular, we see that males (type-m individuals) typically associate a given emotion with darker colors, while females (type-f individuals) associate it with brighter colors. A similar trend was observed with older people and associations to lighter colors. Moreover, through our classification matrix, we identify which colors have weak associations to emotions and which colors are typically confused with other colors.



BRILL

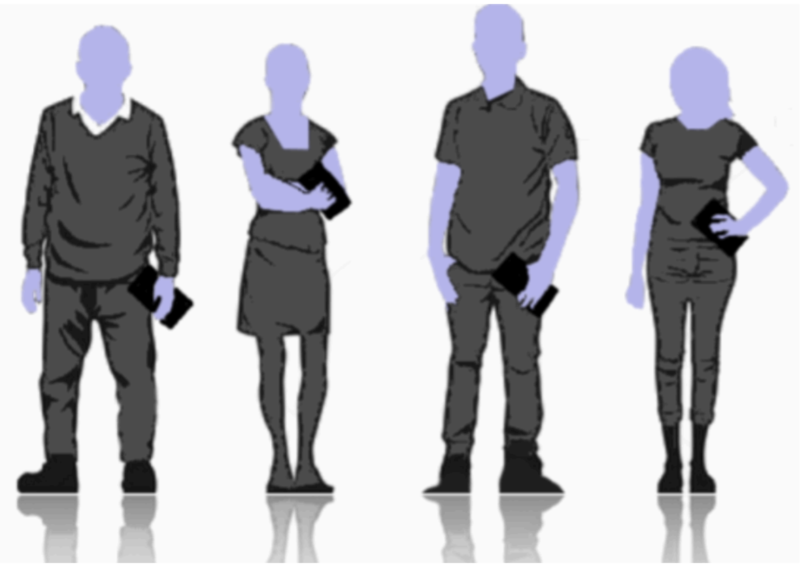
Behaviour (2018) DOI:10.1163/1568539X-00003496

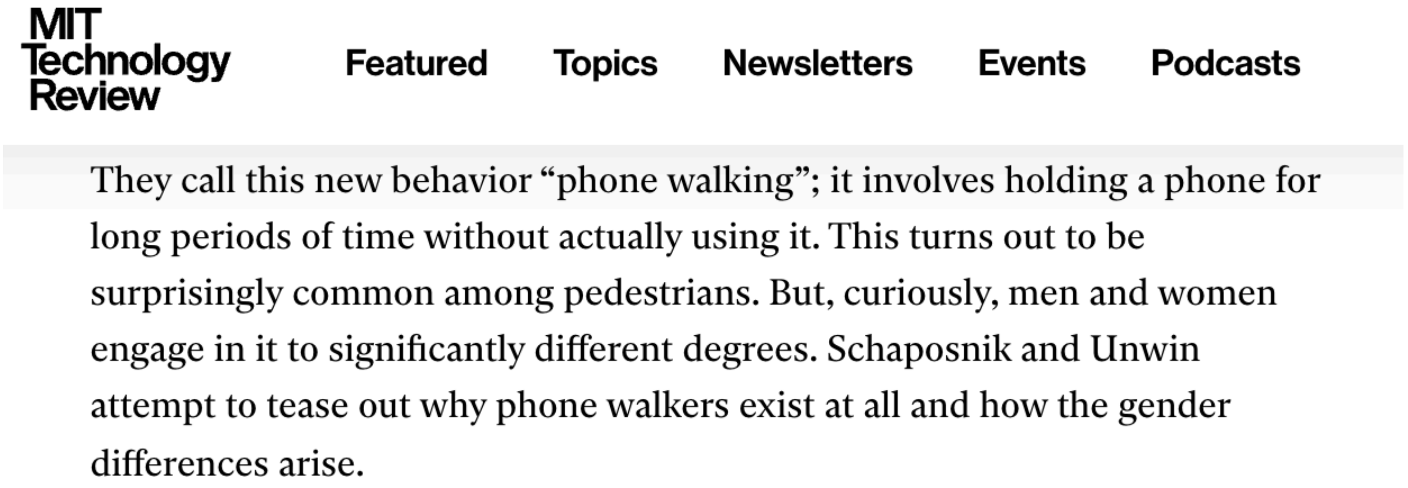
Behaviour
brill.com/beh

The phone walkers: a study of human dependence on inactive mobile devices

Laura P. Schaposnik * **and James Unwin**
University of Illinois at Chicago, Chicago, IL 60647, USA
*Corresponding author's e-mail address: schapos@uic.edu

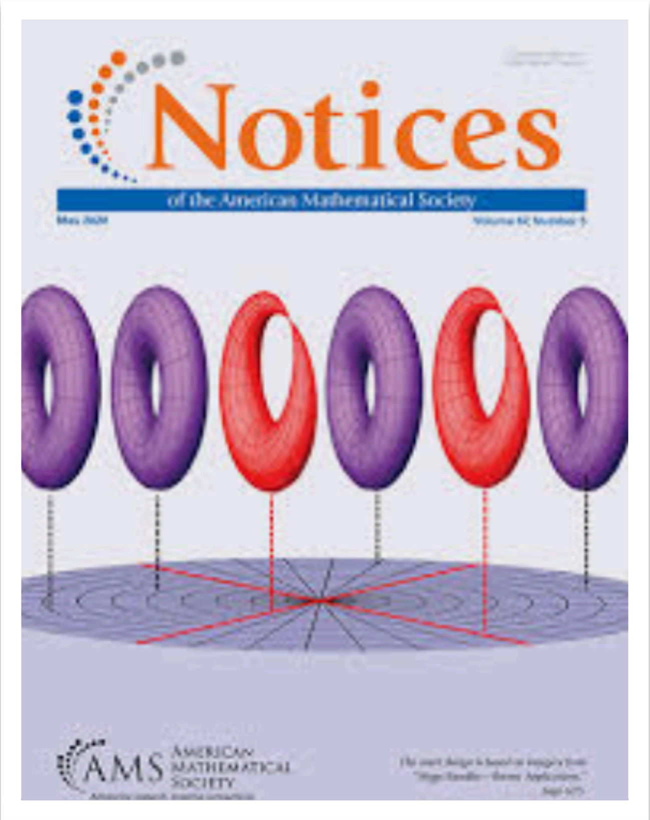
Received 14 October 2017; initial decision 28 January 2018; revised 23 April 2018; accepted 23 April 2018

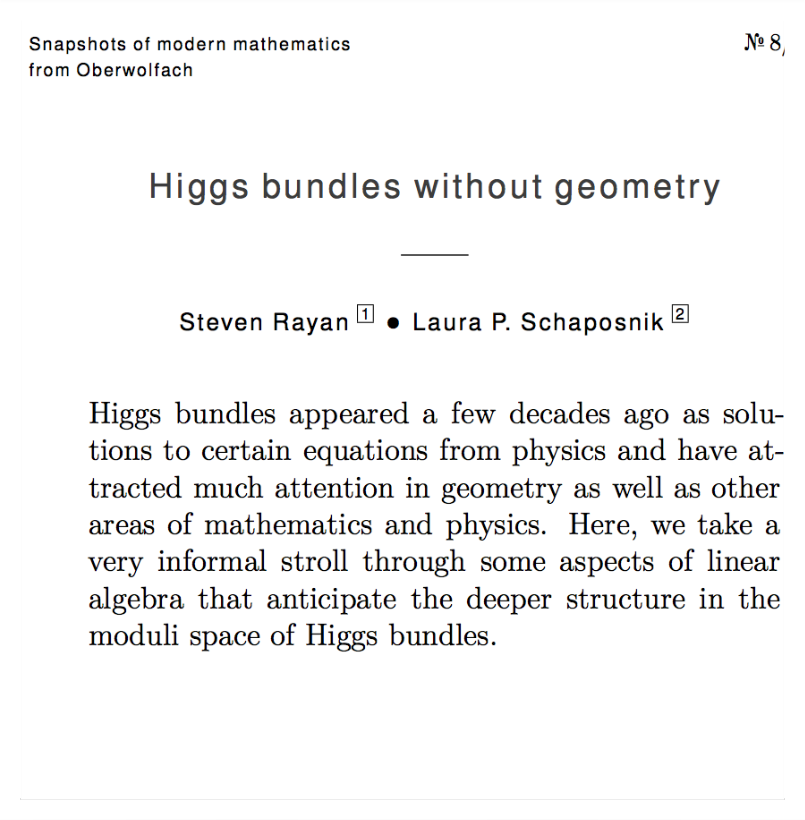




They call this new behavior “phone walking”; it involves holding a phone for long periods of time without actually using it. This turns out to be surprisingly common among pedestrians. But, curiously, men and women engage in it to significantly different degrees. Schaposnik and Unwin attempt to tease out why phone walkers exist at all and how the gender differences arise.

ILLUSTRATING MATHEMATICS





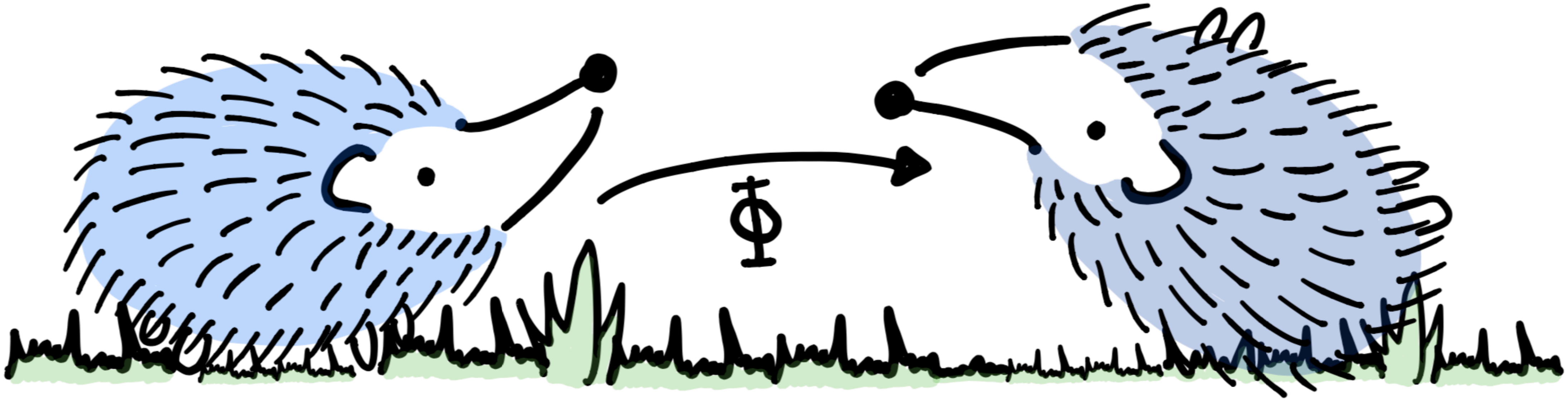
Snapshots of modern mathematics
from Oberwolfach

№ 8,

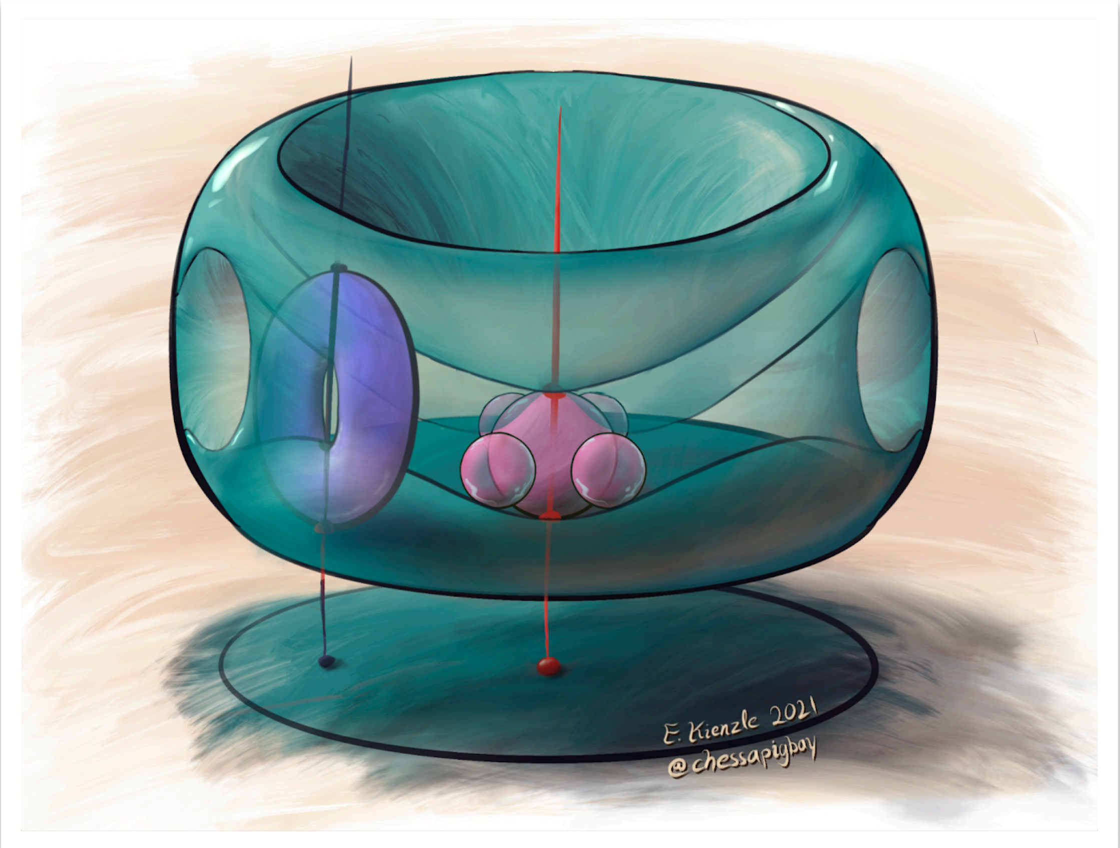
Higgs bundles without geometry

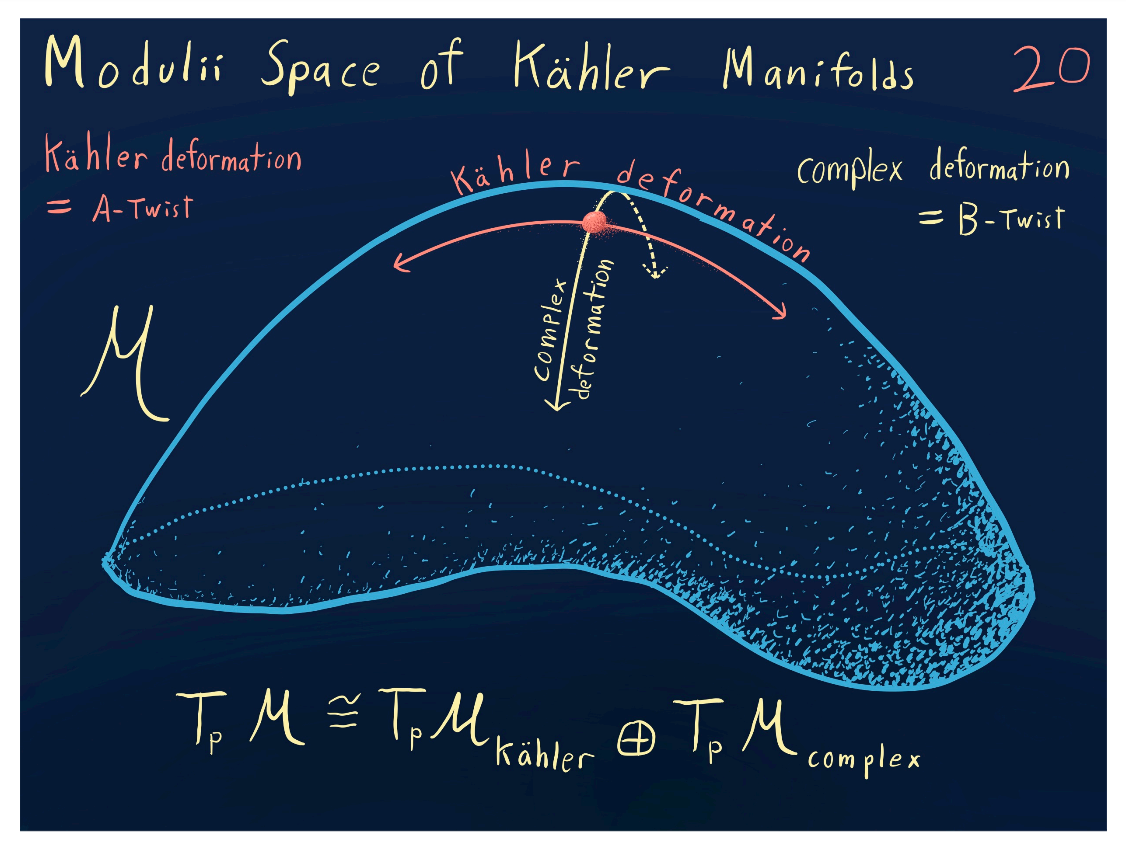
Steven Rayan [1] • Laura P. Schaposnik [2]

Higgs bundles appeared a few decades ago as solutions to certain equations from physics and have attracted much attention in geometry as well as other areas of mathematics and physics. Here, we take a very informal stroll through some aspects of linear algebra that anticipate the deeper structure in the moduli space of Higgs bundles.

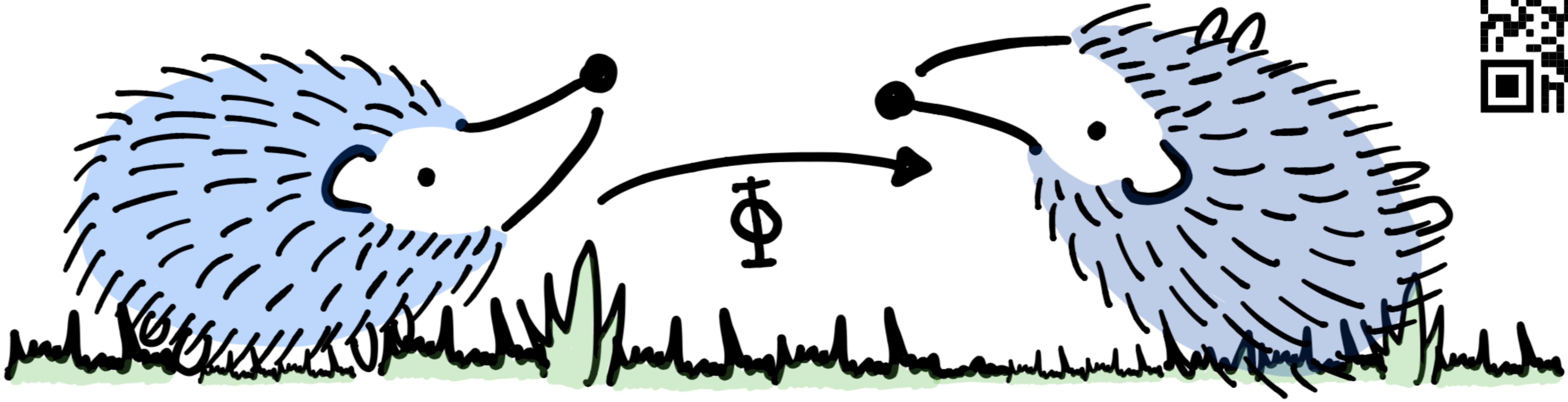


ILLUSTRATING MATHEMATICS





Check out **Elliot Kienzle's** website https://chessapig.github.io/gallery/



Ene's magical adventures...

To teach maths and science words to young children in Spanish and English

Ene siguió trepando,
y a los monitos encontó jugando.

They all played and took a little rest,
and then Ene was given directions to the nest.

En su jardín y entre las hormigas Ene encontró muchas formas matemáticas!

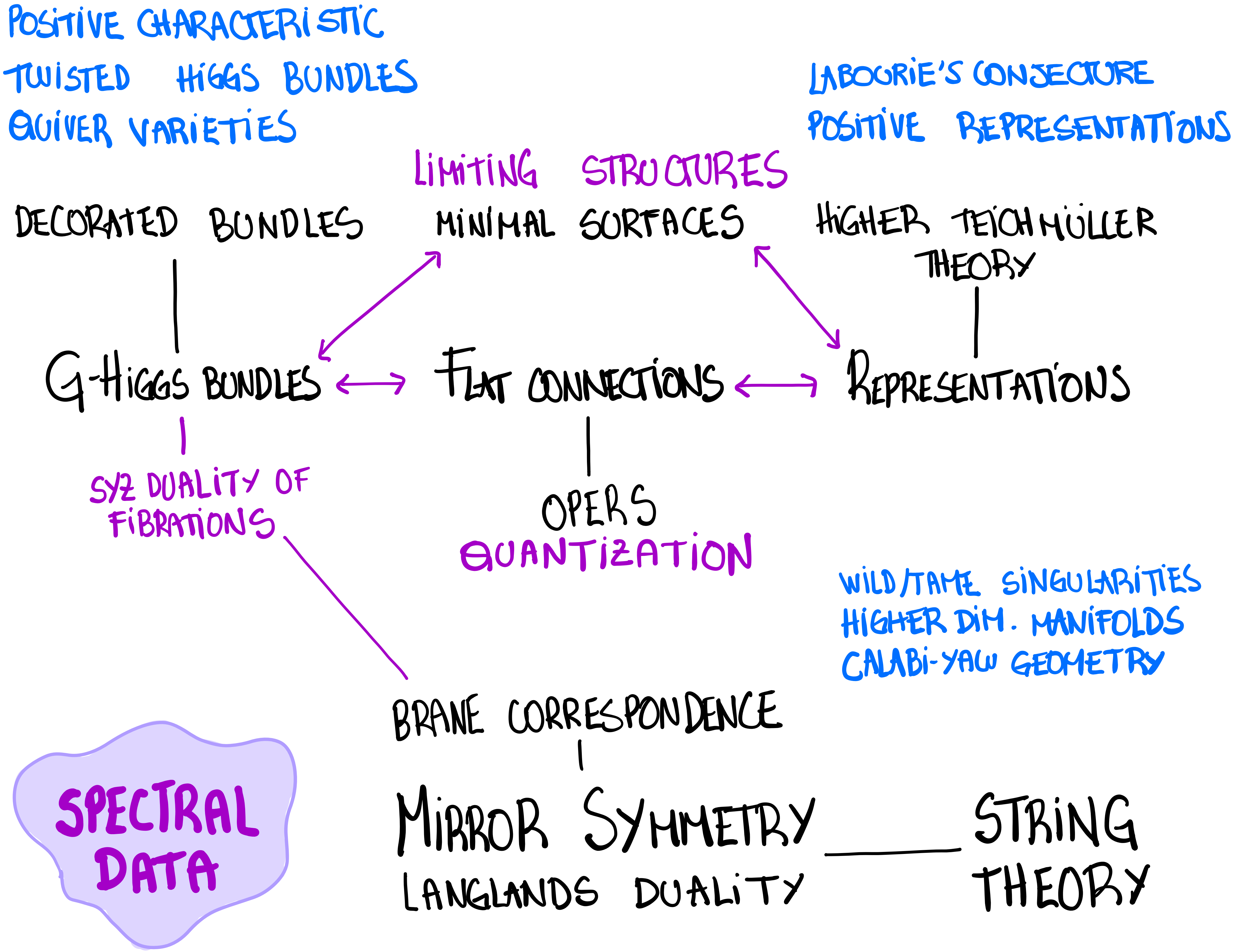
In the garden and inbetween the ants Ene found many mathematical shapes!

Formas en 3D	3D shapes
Una esfera	A sphere
Una cinta de Moebious	A Moebious band
Un ángulo agudo	An acute angle
Una cuerda desanudada	The unknot
Los anillos Borromeanos	Borromean rings
Un campo vectorial	A vector bundle
Un cono	A cone
Simetría	Symmetry
Un cuadrado	A square
Un cubo	A cube
Un cilindro	A cylinder
Una pirámide	A pyramid
Reflexión	Reflection
El símbolo del número Pi	The symbol for Pi
Un toro	A torus
Una superficie de genero 2	A genus 2 surface
Una botella de Klein	A Klein bottle

Laura Schaposnik – University of Illinois at Chicago

GRACIAS!

LAURASCHAPOSNIK.COM



POSITIVE CHARACTERISTIC
TWISTED HIGGS BUNDLES
QUIVER VARIETIES

LABOURIE'S CONJECTURE
POSITIVE REPRESENTATIONS

LIMITING STRUCTURES

DECORATED BUNDLES

MINIMAL SURFACES

HIGHER TEICHMÜLLER THEORY

G-HIGGS BUNDLES

FLAT CONNECTIONS

REPRESENTATIONS

SYZ DUALITY OF FIBRATIONS

OPERS

QUANTIZATION

WILD/TAME SINGULARITIES
HIGHER DIM. MANIFOLDS
CALABI-YAU GEOMETRY

BRANE CORRESPONDENCE

MIRROR SYMMETRY
LANGLANDS DUALITY

STRING THEORY

SPECTRAL DATA