Introduction to Magnetic Monopoles

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MSRI Introductory Workshop: Analytic and Geometric Aspects of Gauge Theory, 2022

- Consider a $\mathrm{SU}(2)$ connection on $\mathbb{C}^2$ bundle $E \to \mathbb{R}^4$ of the form

$$\mathbb{A} = \Phi\, dx_0 + A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3,$$

  with $\Phi, A_1, A_2, A_3 \in \Gamma\big(\, End(E)\big)$ independent of x_0.
- The anti self dual equation $-F_{\mathbb{A}} = \star_4 F_{\mathbb{A}}$ reduces to the *Bogomolny equation*

$$d_A \Phi = \star_3 F_A \tag{1}$$

  on $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3), for the connection $A = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3$ and "Higgs field" $\Phi \in \Gamma(End(E))$.
- *Magnetic monopoles* are solutions of (1) up to gauge equivalence.
- They are local minimizers of the Yang-Mills-Higgs action/energy

$$YMH(\Phi, A) = \frac{1}{2} \int_{\mathbb{R}^3} \|F_A\|^2 + \|d_A \Phi\|^2$$

Motivation: Higgs mechanism

- In quantum Yang-Mills, particles associated to A are massless. Mass arises from terms like $m^2 Q(A)$ in the action where Q is quadratic, but such terms are not gauge invariant.
- The *Higgs mechanism* is the introduction of a scalar (Higgs) field Φ with action of the form

$$\frac{1}{2}\int_{\mathbb{R}^{3+1}} \|F_A\|^2 + \|D_A\Phi\|^2 + V(\Phi) \tag{2}$$

  with V a potential, e.g., of the form $V(\Phi) = \frac{1}{4}\lambda(1 - |\Phi|^2)^2$.
- Effective quadratic terms in A arise through coupling to Φ near the minima of the potential.
- The *Bogomolny-Prasad-Sommerfield (BPS) limit* $\lambda \to 0$, recovers the action (in Euclidean case) for monopoles.
- Monopoles are static (time independent) solutions to the dynamical ($3+1$ dimensional) Euler-Lagrange equations for (2) with minimal energy.

Basic properties: mass and charge

- Solutions on compact 3 manifolds are trivial: $\star F_A = d_A\Phi = 0$, consisting of flat $\mathrm{SU}(2)$ connections (if $\Phi \equiv 0$) or flat $\mathrm{U}(1)$ connections (if $\Phi \not\equiv 0$).
- Not conformally invariant, so solutions on $\mathbb{R}^3$ not obtained from compactification to S^3, and require attention to boundary conditions "at infinity", as $r \to +\infty$. Also, there is no "bubbling" as in 4D Y-M.
- Finite energy $\frac{1}{2}\int \|F_A\|^2 + \|d_A\Phi\|^2 < \infty$ on $\mathbb{R}^3$ implies that

$$\lim_{r\to\infty} |\Phi| = m$$

  exists; this is the **mass**, which we will typically normalize to 1.
- Then

$$\lim_{r\to\infty} \Phi : S^2_\infty \to S^2 \subset \mathfrak{su}(2) \cong \mathbb{R}^3$$

  has a homotopy class $k \in \mathbb{Z}$; this is the **charge**. Equivalently, the trivial bundle E splits as $L \oplus L^{-1}$ over S^2_∞ into (eigen) line bundles of Φ, with $c_1(L) = k \in H^2(S^2_\infty; \mathbb{Z}) = \mathbb{Z}$.
- Integration by parts gives

$$\frac{1}{2}\int \|F_A\|^2 + \|d_A\Phi\|^2 = \int \|\star F_A - d_A\Phi\|^2 \pm 4\pi k$$

  So for $k \geq 0$ minimizers (on finite energy components) are given by $\star F_A = d_A\Phi$.

- In physics parlance, where $\Phi \neq 0$ (in particular at infinity), the $\mathrm{SU}(2)$ symmetry is "broken" to $\mathrm{U}(1)$ (the centralizer of Φ).
- Field components along Φ approximate a $\mathrm{U}(1)$ monopole with charge k, and field components along $\Phi^{\perp}$ decay exponentially.
- The $\mathrm{U}(1)$ monpole equations $\star F_A = d\phi \in \Omega^1(\mathbb{R}^3)$ describe a magnetic field $B = \star F_A$ generated by a scalar potential ϕ.
- Dirac found solutions on $\mathbb{R}^3 \setminus 0$ with $\phi = k/r$, with A a connection the line bundle of degree k. The charge k is therefore quantized for topological reasons, even in the classical theory.
- While $\mathrm{U}(1)$ monopoles have singularities and infinite energy, $\mathrm{SU}(2)$ monopoles "look like" magnetic monopoles for $r \gg 1$ yet have finite energy and smooth field configurations.

- Bogomolny, Prasad, and Sommerfield found an explicit solution of charge 1 of the form

$$\Phi = \sum_j \frac{i}{2}\left(\frac{1}{\tanh r} - \frac{1}{r}\right)\widehat{x}_j\,\sigma_j$$
$$A = \sum_{j,k,l} \frac{i}{2}\left(\frac{1}{\sinh r} - \frac{1}{r}\right)\epsilon_{jkl}\widehat{x}_j\,\sigma_k\,dx_l$$

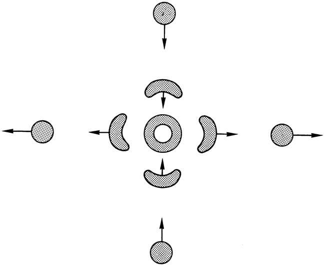
  where $\widehat{x}_j = x_j/r$ and σ_j are Pauli matrices. Note $|\Phi| \leq 1$ and Φ has a unique zero at the origin.
- Translation in $\mathbb{R}^3$ gives a family of solutions centered at any point.
- Gauge transformation by $\exp(t\Phi)$ for $t \in \mathbb{R}/2\pi\mathbb{Z}$ gives an additional circle action, giving a family of solutions parameterized by $\mathbb{R}^3 \times S^1$
- Monopoles of charge $k \geq 2$ are intractible to write down explicitly, though some special solutions (with various symmetries) can be well-described numerically, or as other equivalent objects (to come).

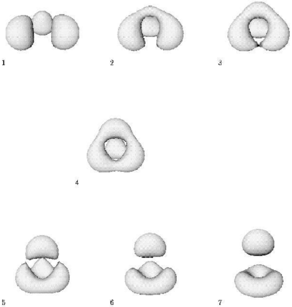
Higher charge monopoles from gluing

- While $\mathrm{U}(1)$ monopoles (satisfying a linear equation) may be superposed to obtain higher charge monopoles, the Bogomolny equation for $\mathrm{SU}(2)$ is nonlinear so superposition doesn't work.
- However, if the constituent monopoles are "widely separated", the nonlinearities are small and a gluing theorem of Taubes [20] shows that superpositions may be perturbed to solutions.
- This gives existence of charge k monopoles for all $k \geq 0$, as widely separated charge 1 monopoles.
- Monopoles are "solitons", field configurations behaving roughly as particles, with the charge k as a particle number.
- The particle picture becomes invalid when the points become close to each other.

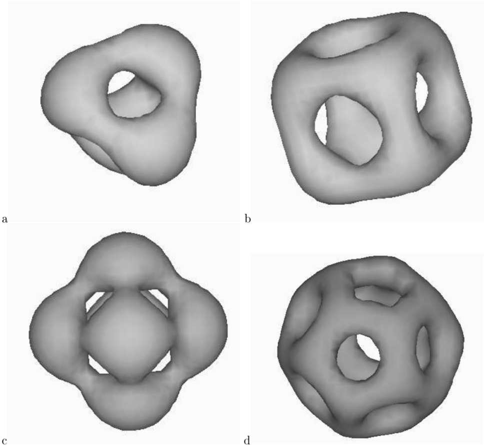
Pictures

- $k = 2$ and $k = 3$ “scattering” processes from [2] and [33]





- Monopoles with platonic symmetries from [33]



- Charges 3, 4, 5, and 7, respectively.

- The **moduli space**, $\mathcal{M}_k$, of monopoles of charge k is the set of charge k solutions to the Bogomolny equations up to gauge equivalence.
- With $\mathcal{A}_k = \left\{(A,\Phi) : YMH(A,\Phi) < \infty, [\Phi]_{S^2_\infty} = k\right\}$ the finite energy, charge k, component of the configuration space, $\mathcal{G} = \Omega^0(\mathbb{R}^3; Aut(E))$ the gauge group and and

$$\mathcal{B} : \mathcal{A}_k \to \Omega^1(\mathbb{R}^3; End(E)), \qquad \mathcal{B}(A,\Phi) = \star F_A - d_A\Phi,$$

the moduli space is the quotient

$$\mathcal{M}_k = \mathcal{B}^{-1}(0)/\mathcal{G}$$

- Parameter counting, and later a (nontrival) index theorem proved by Taubes shows that $\dim \mathcal{M}_k = 4k - 1$.
- Deformations are unobstructed, so $\mathcal{M}_k$ is smooth.
- $\mathcal{M}_k$ is also non-compact, not because of bubbling as in 4D Yang-Mills, but because of the region of the moduli space in which well-separated 1-monopoles may become arbitrarily far apart.

- To get a natural metric, an enlarged moduli space of "framed" monopoles is in order.
- A **framing** of a k-monopole is a choice of isomorphism $E \cong H^k \oplus H^{-k}$ on S^2_∞ where H is the fundamental line bundle on S^2 with its standard connection.
- A **framed monopole** is a solution of the form $(\Phi_0 + \phi, A_0 + a)$ where

$$\Phi_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(1 - \frac{k}{2r}\right)$$

  and A_0 is associated to the standard connection on H^k with curvature $F_{A_0} = k\, Vol_{S^2}$, and $|(\phi, a)| = O(r^{-2})$.
- The framed moduli space $\widetilde{\mathcal{M}}_k$ is the space of framed solutions modulo the set $\mathcal{G}_0$ of gauge transformations which limit to the identity as $r \to \infty$, preserving the framing.
- $\widetilde{\mathcal{M}}_k \to \mathcal{M}_k$ is a circle bundle, with circle action given by the gauge transformations of the form $\exp(t\Phi)$ (which are not in $\mathcal{G}_0$). Thus $\dim(\widetilde{\mathcal{M}}_k) = 4k$.

L^2 metric on $\widetilde{\mathcal{M}}_k$

- Infinitesimal variations are square integrable, thus $T_{(A,\Phi)}\widetilde{\mathcal{M}}_k$ identified with (ϕ, a) in L^2 such that

$$\star d_A a - d_A\phi + [\Phi, a] = 0 \qquad \text{linearization of Bogomolny}$$
$$\star d_A \star a + [\Phi, \phi] = 0 \qquad \text{Coulomb gauge}$$

- The L^2 metric given by

$$g\big((\phi, a), (\psi, b)\big) = \int_{\mathbb{R}^3} (\phi, \psi) + (a, b)$$

is Riemannian and complete by results of Taubes and Uhlenbeck.
- $\widetilde{\mathcal{M}}_1 = \mathbb{R}^3 \times S^1$ isometrically, and for $k \geq 2$, $\widetilde{\mathcal{M}}_k$ splits as a Riemannian product $(\mathbb{R}^3 \times S^1) \times \mathcal{M}_k^0$, up to k-fold cover, where $\mathcal{M}_k^0$ is the *reduced moduli space* of centered monopoles. In the "widely separated 1-monopole" region, g is exponentially asymptotic to a model metric of Gibbons and Manton [13] by a result of Bielawski [3].
- $T\widetilde{\mathcal{M}}_k$ has a quaternionic action under the identification

$$\phi + a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3 \quad \leftrightarrow \quad \phi + a_1 I + a_2 J + a_3 K$$

- Morever, the complex structures I, J, K are integrable and $\widetilde{\mathcal{M}}_k$ (and $\mathcal{M}_k^0$) is a *hyperKähler* manifold.

- Recall a hyperKähler manifold M^{4n} has 3 integrable complex structures (I, J, K) satisfying quaternionic relations and a Riemannian metric g such that each two form $\omega_\bullet = g(\cdot, \bullet\cdot)$ is closed, for $\bullet \in \{I, J, K\}$.
- In fact, there is a 2-sphere of integrable complex structures, $\left\{aI + bJ + cK : a^2 + b^2 + c^2 = 1\right\}$.
- Equivalently, M^{4n} has holonomy group $\mathrm{Sp}(n)$, and in particular is Ricci flat (Calabi-Yau).
- To see that $\widetilde{\mathcal{M}}_k$ is hyperKähler, follow Hitchin [17] to exhibit $\widetilde{\mathcal{M}}_k$ as an infinite dimensional *hyperKähler quotient*

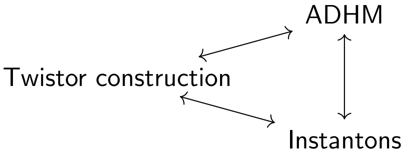
$$\widetilde{\mathcal{M}}_k = \widetilde{\mathcal{A}}_k /// \mathcal{G}_0 = \mathcal{B}^{-1}(0)/\mathcal{G}_0$$

of the affine hyperKähler space $\widetilde{\mathcal{A}}_k \ni (\Phi, A) \leftrightarrow \Phi + A_1 I + A_2 J + A_3 K$ by the gauge group, where

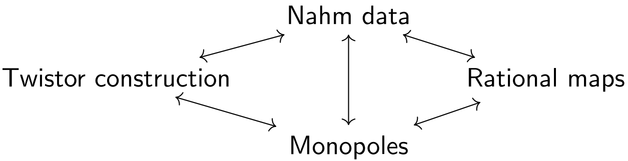
$$\mathcal{B}(\Phi, A) = \star F_A - d_A \Phi \in Lie(\mathcal{G}_0) \otimes Im\,\mathbb{H}$$

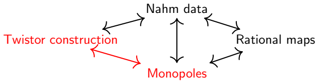
is the hyperKähler moment map with $(dx_1, dx_2, dx_3) \in Im\,\mathbb{H}$

- Instantons on $\mathbb{R}^4$ have equivalent descriptions:

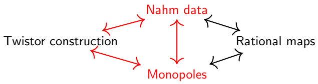


- Analogously, there are equivalent descriptions for monopoles:





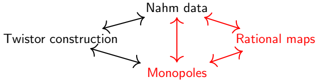
- The twistor space for monopoles is the complex surface $T\mathbb{C}P^1$, viewed as the space of oriented lines in $\mathbb{R}^3$.
- By a result of Hitchin [15] (based on a construction of Ward as extended by Corrigan and Goddard), Monopoles are equivalent to certain complex vector bundles with holomorphic sections on twistor space, or equivalently certain *spectral curves*, the zero sets of the holomorphic sections.
- The spectral curve for a monopole (Φ, A) consists of those lines in $\mathbb{R}^3$ along which the ODE $(\nabla_A - i\Phi)s = 0$ has L^2 solutions (decay at both ends).



- **Nahm data** is a set of three $k \times k$ matrix-valued analytic functions T_1, T_2, T_3 on the interval $(0, 2)$ satisfying *Nahm's equations*

$$\frac{d}{ds} T_i + \frac{1}{2} \sum \epsilon_{ijk} [T_j, T_k] = 0, \quad T_i^*(s) = -T_i(s), \quad T_i(2-s) = T_i(s)^T$$

  with simple poles at $s = 0, 2$ with residues forming an irreducible representation of $\mathfrak{su}(2)$.
- Like the Bogomolny equation, Nahm's equations are a dimensional reduction of the instanton equations from $\mathbb{R}^4$ to $\mathbb{R}^1$.
- The *Nahm transform* gives a bijection [16, 29] between $O(k, \mathbb{C})$ gauge equivalence classes of such solutions and charge k monopoles. It is the seminal example of a more general Nahm transform [21], a "Fourier transform" between Yang-Mills objects on $\mathbb{R}^4/\Lambda$ and a dual $(\mathbb{R}^4)^*/\Lambda^*$.
- The moduli space of Nahm data obtains a hyperKähler metric from a hyperKähler quotient construction, and the Nahm transform is an isometry [29].



- A result of Donaldson [8] identifies Nahm data with the space $R_k(\mathbb{C}P^1)$ of degree k rational maps $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $f(\infty) = 0$, i.e. of the form
$$f(z) = \frac{p(z)}{q(z)} = \frac{a_{k-1}z^{k-1} + \cdots + a_0}{z^k + b_{k-1}z^{k-1} + \cdots + b_0}$$
where $p(z)$ and $q(z)$ are coprime.
- Hurtubise [19] described the direct connection between monopoles and rational maps in terms of the scattering map for the linear operator $(\nabla_A - i\Phi)$ along lines in $\mathbb{R}^3$.
- This leads to a detailed understanding of the topology of $\widetilde{\mathcal{M}}_k \cong R_k(\mathbb{C}P^1)$ [32], but the L^2 metric is not directly apparent.
- Atiyah and Hitchin in [2] use this representation to get a remarkably complete understanding of the reduced 2-monopole space $\mathcal{M}_2^0$.

Open problems and current research

- Sen conjecture
- Higher rank gauge groups
- Monopoles on non-$\mathbb{R}^3$
- Magnetic bag conjecture

- S-duality in SUSY QFT leads to a physical prediction for the L^2-cohomology of the reduced moduli spaces $\mathcal{M}_k^0$ (or rather their universal k-fold covers $\widehat{\mathcal{M}_k^0}$), known as *Sen's Conjecture*:

$$\mathscr{H}^i(\widehat{\mathcal{M}_k^0}) = \begin{cases} \mathbb{C}^{|\mathbb{Z}_k^*|} & i = 2k-2 \\ 0 & \text{otherwise} \end{cases}$$

- Here $\mathscr{H}^i$ denotes the space of i forms which are L^2 integrable with respect to the hyperKähler metric.
- By general results of Hitchin concerning L^2 cohomology of hyperKähler quotients, the vanishing outside of middle degree is known, and in [32] Segal and Selby computed $Im\left(H_c^\bullet(\widehat{\mathcal{M}_k^0}) \to H^\bullet(\widehat{\mathcal{M}_k^0})\right)$, which is consistent with the conjecture.
- Outstanding issue is to show that L^2 harmonic forms have appropriate decay, the subject of my ongoing work with Singer and Rochon.

- Singer and I [26, 12] construct a compactification of $\mathcal{M}_k^0$ to a manifold with corners, the boundary hypersurfaces of which encode limiting configurations of widely separated monopole "clusters" of charges k_i where $k = \sum k_i$.
- The metric has well-behaved asymptotics up to each hypersurface, and is an example of a "quasi-fibered boundary" (QFB) metric, generalizing the class of ALF metrics in the classification of 4d hyperKähler manifolds.
- Rochon and I developed tools [25] (a pseudodifferential operator calculus) for analysis on QFB manifolds, with some initial results on L^2 Hodge theory in this setting which include Sen for $k = 3$ [24].
- I expect similar considerations should apply to Hitchin systems, at least in simple cases.

- We can consider monopoles for compact gauge groups other than $\mathrm{SU}(2)$.
- The general rational map description is in Jarvis [22], Nahm data description by Hurtubise and Murray [18] and more recently Charbonneau and Nagy [6].
- Many questions about metrics on these moduli spaces are open.
- The mass and charge are replaced by elements $\mu, \kappa \in \mathfrak{g}$ with $[\mu, \kappa] = 0$.
- How the symmetry is broken at infinity becomes a complex question. The best case is "maximal symmetry breaking", where $\Phi\big|_{S^2_\infty}$ has distinct eigenvalues. Non-maximal symmetry breaking is more subtle.
- Here the moduli space is *stratified*, and while the metric on the total space may not be hyperKähler, Murray and Singer have a conjecture [28] that the metric on certain strata (with dimensions in multiples of 4) is hyperKähler.

- Reductions from $\mathbb{R}^4$ include monopoles (with singularities) on:
  - $\mathbb{S}^1 \times \mathbb{R}^2$, which have been constructed by Foscolo [11], which following a program Cherkis and Kapustin [7] include families of ALG type hyperKähler manifolds
  - $\mathbb{T}^2 \times \mathbb{R}$, so-called "monowalls" by Cherkis and Ward.
  - $\mathbb{T}^3$, studied via Nahm transform by Charbonneau and Hurtubise [5].
- Certain moduli spaces of the above should be complete and hyperKähler.
- Monopoles on asymptotically hyperbolic manifolds were studied by Braam [4] and earlier (on $\mathbb{H}^3$) by Atiyah [1].
- Monpoles with singularities on compact 3-manifolds have been considered by Pauly [31] (virtual dimension)
- There has been some work on monopoles over *asymptotically conic* (AC) 3-manifolds other than $\mathbb{R}^3$ by myself [23], Oliveira, and Fadel [30, 10, 9] including the (virtual) dimension, the construction of some smooth families, and analysis of asymptotics.

- A **magnetic bag** consists of a magnetic (i.e., $\mathrm{U}(1)$) field arising from magnetic charge uniformly distributed over a closed surface (the "bag") in $\mathbb{R}^3$, thought of as an infinitesimally thin shell. The field vanishes inside the bag, and is Coulomb-like at large distances.
- Bolognesi conjectured that in the large charge limit, as $k \to \infty$, there are sequences of $\mathrm{SU}(2)$ monopoles converging to given magnetic bags.
- More quantitatively, this involves the question of just how concentrated large charge monopoles can be.
- Hueristically, maximally concentrated monopoles have a region of size $O(k)$ in which Φ is "small", a thin shell region of width $O(k^{1/2})$ where the Yang-Mills-Higgs energy is concentrated, and outside of this the fields are approximately abelian. This is bourne out by results of Taubes [34]
- Harland [14] has work connecting the large k limit of Nahm data to magnetic bags.
- A survey by Manton [27] discusses related objects, so-called monopole "planets" and "galaxies".

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