

Khovanov Homology  
and the  
Involutive Heegaard Floer Homology  
of Branched Double Covers

Melissa Zhang

UGA  $\longrightarrow$  MSRI/SLMath  $\longrightarrow$  UC Davis

# Roadmap

- Khovanov Homology  $Kh(L)$ ,  $\widehat{Kh}(L)$  [Khovanov]
- Heegaard Floer Homology  $\widehat{HF}(Y^3)$  [Ozsváth-Szabó]
- Spectral Sequence  $\widehat{Kh}(L) \Rightarrow \widehat{HF}(\Sigma(L))$  [Oz-Sz]
- Involutive HF Homology  $\widehat{HFI}(Y^3)$  [Hendricks-Manolescu]
- Involutive Spectral Sequence [Alishahi-Truong-Z., WIP]  
(see work of Francesco Lin)

## Tools

- Bordered Floer Homology [Lipshitz-Ozsváth-Thurston]  
& Bordered version of the spectral sequence  $\widehat{Kh} \Rightarrow \widehat{HF}(\Sigma)$
- Involutive Bordered Floer homology [Hendricks-Lipshitz]

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## Tools

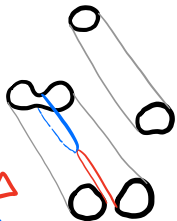
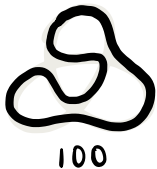
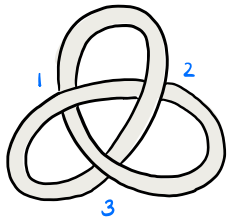
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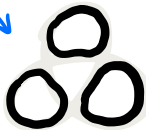
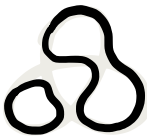
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$A \otimes A$



000

010

101

111

$M$

$A$

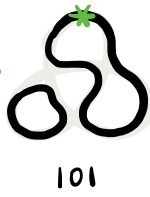
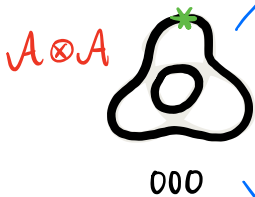
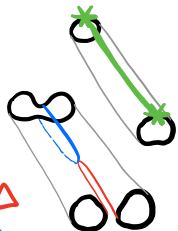
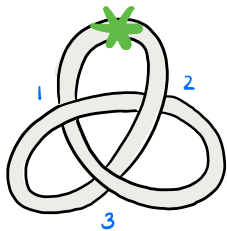


001

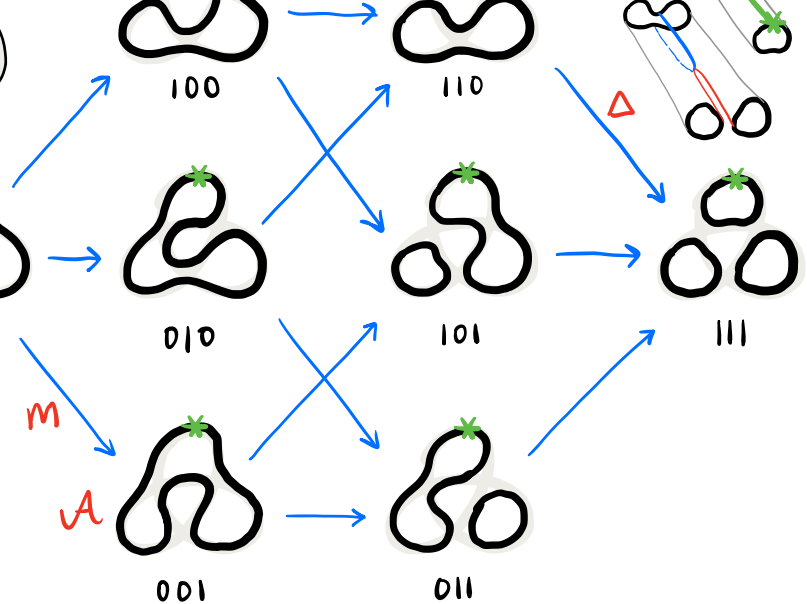
011

$$A = \frac{\mathbb{F}[X]}{(X^2)}$$





$$\mathcal{A} = \frac{\mathbb{F}[X]}{(X^2)}$$



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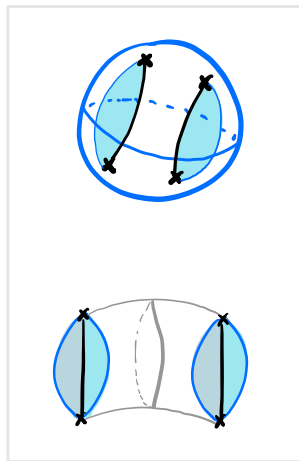
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$$\Sigma(\text{torus}) = \Sigma(\text{figure-eight})$$

$$= \Sigma(\text{two spheres}) \cup \Sigma(\text{two spheres})$$

$$= \text{torus} \cup \text{torus}$$

$$= S^1 \times S^2$$



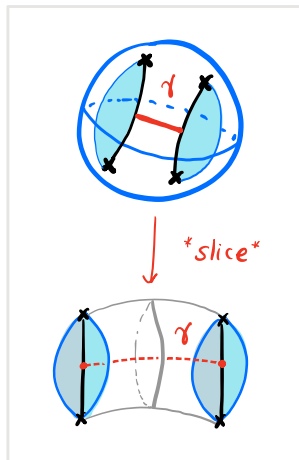


$$\Sigma \left( \text{triple point} \right) = \Sigma \left( \text{figure-eight} \right)$$

$$= \Sigma \left( \text{two spheres} \right) \cup \Sigma \left( \text{two spheres} \right)$$

$$= \left( \text{torus} \right) \cup \left( \text{torus} \right)$$

$$= (S^1 \times S^2, \tilde{\gamma})$$

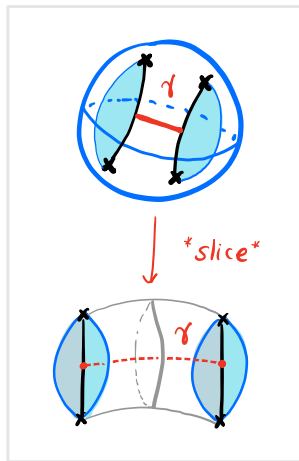


$$\Sigma \left( \text{[trefoil knot with green star and red dot]} \right) = \Sigma \left( \text{[figure-eight knot with red dot]} \right)$$

$$= \Sigma \left( \text{[sphere with red curve } \gamma \text{ and four black dots]} \right) \cup \Sigma \left( \text{[sphere with two black dots]} \right)$$

$$= \text{[torus with red dashed curve } \tilde{\gamma} \text{ and four black dots]} \cup \text{[torus with one black dot]}$$

$$= (S^1 \times S^2, \tilde{\gamma}) \xrightarrow{\text{(-1)-surgery on } \tilde{\gamma}} \Sigma \left( \text{[trefoil knot with green star]} \right) = S^3$$



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Cone

$$\widehat{CF}(\#^k S' \times S^2)$$

$$Q \cdot (\text{id} + \iota) \downarrow$$

$$Q \cdot \widehat{CF}(\#^k S' \times S^2)$$

$$= \widehat{CFI}(\#^k S' \times S^2)$$

homology  
→

$$L \simeq \boxtimes^k L_{S' \times S^2}$$

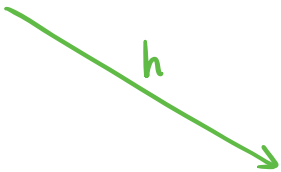
$$\widehat{HF}(\#^k S' \times S^2)$$

⊕

$$Q \cdot \widehat{HF}(\#^k S' \times S^2)$$

$(E', d')$ :

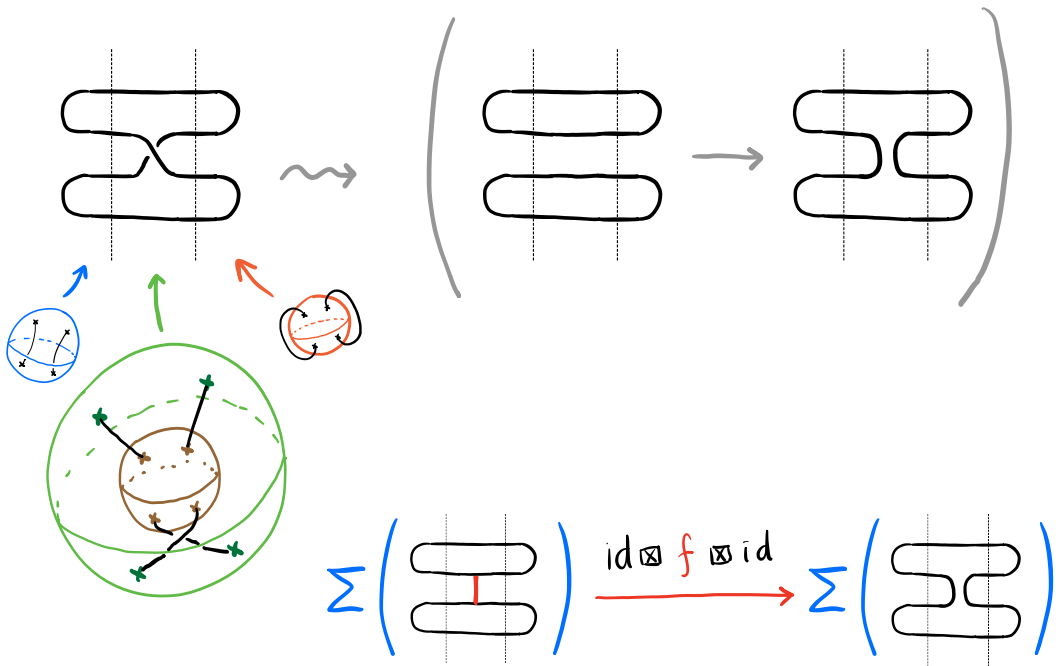
$$\widehat{HF}(\#^k S' \times S^2) \xrightarrow{s} \widehat{HF}(\#^{k-1} S' \times S^2)$$



$$Q \cdot \widehat{HF}(\#^k S' \times S^2) \xrightarrow{s} Q \cdot \widehat{HF}(\#^{k-1} S' \times S^2)$$

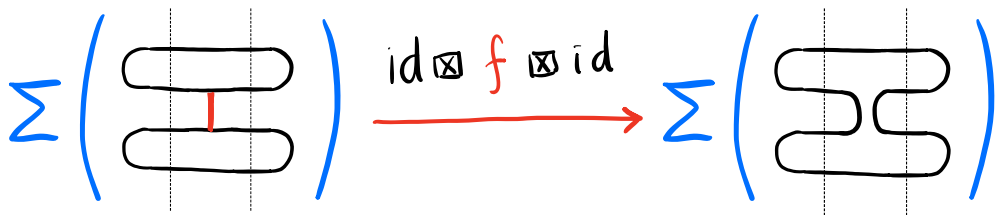
$$\underbrace{\hspace{15em}}_{\widehat{HFI}(\#^k S' \times S^2)}$$

$$\underbrace{\hspace{15em}}_{\widehat{HFI}(\#^{k+1} S' \times S^2)}$$



$$\widehat{\text{CFD}}(H_\infty) \quad \widehat{\text{CFD}}(H_{-1})$$

$$\widehat{\text{CFA}}(\mathcal{H}_L) \boxtimes \widehat{\text{CFDAA}}(\mathcal{H}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_R) \quad \widehat{\text{CFA}}(\mathcal{H}_L) \boxtimes \widehat{\text{CFDAA}}(\mathcal{H}) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_R)$$



$\mathcal{H}$  = bordered diagram for complement of  $\mathcal{F}$

$$\begin{array}{ccc}
 [\mathcal{H}_\infty] & \xrightarrow{\quad\quad\quad} & [\mathcal{H}_{-1}] \\
 \begin{array}{c} \text{D} \\ | \\ \text{A} \\ | \\ [\mathcal{H}] \end{array} & & \begin{array}{c} \text{D} \\ | \\ \text{A} \\ | \\ [\mathcal{H}] \end{array} \\
 [\mathcal{H}_L] \text{A} \text{---} \text{D} \text{---} & & [\mathcal{H}_L] \text{A} \text{---} \text{D} \text{---} \\
 & & [\mathcal{H}_R]
 \end{array}$$

$$\Sigma \left( \begin{array}{c} \text{---} \\ \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \right) \xrightarrow{\text{id} \boxtimes f \boxtimes \text{id}} \Sigma \left( \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right)$$

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$$\begin{array}{c}
 [\mathcal{H}_\bullet] \\
 \text{D} \\
 | \\
 \text{A} \\
 | \\
 [\mathcal{H}_L] \text{A} \text{---} \text{D} [\mathcal{H}] \text{A} \text{---} \text{D} [\mathcal{H}_R]
 \end{array}
 \quad =: \quad
 \begin{array}{c}
 [\mathcal{H}_\bullet] \\
 \text{D} \\
 | \\
 \text{A} \\
 | \\
 [\mathcal{H}_X]
 \end{array}$$

$\uparrow$   
 exterior of  $\tilde{\gamma}$   
 in  $\#^k S^1 \times S^2$

$$\begin{array}{ccc}
[\mathcal{H}_x] \xrightarrow{A} [\mathcal{H}_\infty] & \longrightarrow & [\mathcal{H}_x] \xrightarrow{A} [\mathcal{H}_{-1}] \\
\downarrow \eta & & \downarrow \eta \\
[\overline{\mathcal{H}}_x] \xrightarrow{A} [\overline{\mathcal{H}}_\infty] & \longrightarrow & [\overline{\mathcal{H}}_x] \xrightarrow{A} [\overline{\mathcal{H}}_{-1}] \\
\downarrow \Omega & & \downarrow \Omega \\
[\overline{\mathcal{H}}_x] \bullet [AZ] \bullet [\overline{\mathcal{H}}_\infty] & \longrightarrow & [\overline{\mathcal{H}}_x] \bullet [AZ] \bullet [\overline{\mathcal{H}}_{-1}] \\
\downarrow \Psi_0 \boxtimes \text{id} \boxtimes \text{id} & & \downarrow \Psi_0 \boxtimes \text{id} \boxtimes \text{id} \\
[\mathcal{H}_x] \bullet [AZ] \bullet [\overline{\mathcal{H}}_\infty] & \longrightarrow & [\mathcal{H}_x] \bullet [AZ] \bullet [\overline{\mathcal{H}}_{-1}] \\
\downarrow \text{id} \boxtimes \Psi_1 & \searrow H & \downarrow \text{id} \boxtimes \Psi_1 \\
[\mathcal{H}_x] \bullet [\overline{\mathcal{H}}_\infty] & \longrightarrow & [\mathcal{H}_x] \bullet [\overline{\mathcal{H}}_{-1}]
\end{array}$$

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