$HF^*(Y, v)$

Azam-Cannizzo-Lee-Liu

ACLL example

Lagrangians

Floer theory

Floer theory of a symplectic fibration: an example

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A non-exact symplectic fibration

Recall: a moment polyhedron Δ defines a toric symplectic manifold via symplectic reduction.

We obtain a family of complex 3-folds Y parametrized by Δ_Ω for symmetric $\Omega > 0 \in M_2(\mathbb{R})$.

Product of toric coordinates defines $v : Y \to \mathbb{C}$ symplectic fibration with

- base $\mathbb{C}$, generic fiber T^4
- singular fiber over 0 with Crit$(v) = \cup$ 3 $\mathbb{CP}^1$'s/ $\sim$

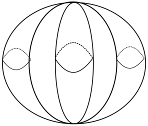


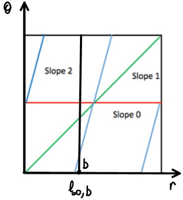
Figure: Critical locus of singular fiber

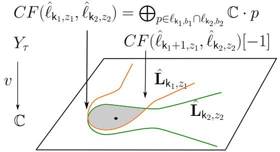
Lagrangians + connection 1 form

Ω gives ω, $B \in H^2(Y)$ *B-field* $\rightsquigarrow B + i\omega = \omega_{\mathbb{C}}$, $\tau = B|_{v^{-1}(-1)} + i\Omega$,
$k \in \mathbb{Z}$ "slope", $b \in \mathbb{R}^2$ "shift", $a \in \mathbb{R}^2$ "connection", $z = a + \tau b$

$\hat{\mathbf{L}}_{\mathsf{k},z} = (\mathbf{L}_{\mathsf{k},b}, \mathcal{E}_a)$: $v(\mathbf{L}_{\mathsf{k},b})$ = U-shape around 0,
Lagr $\mathbf{L}_{\mathsf{k},b}|_{v^{-1}(-1)} = \{(r_1, r_2, \theta_1, \theta_2) \in T^4 | \theta = \mathsf{k}r - b\}$,

$\mathcal{E}_a = (\underline{\mathbb{C}}, \nabla_a)$ s. t. $d\nabla_a = -2\pi i B|_{\mathbf{L}_{\mathsf{k},b}}$,
$\nabla_a|_{v^{-1}(-1)} = d - 2\pi i a \cdot dr$.





$CF(\hat{\mathbf{L}}_{\mathsf{k}_1,z_1}, \hat{\mathbf{L}}_{\mathsf{k}_2,z_2}) \cong CF(\hat{\ell}_{\mathsf{k}_1+1,z_1}, \hat{\ell}_{\mathsf{k}_2,z_2})[-1] \oplus CF(\hat{\ell}_{\mathsf{k}_1,z_1}, \hat{\ell}_{\mathsf{k}_2,z_2})$

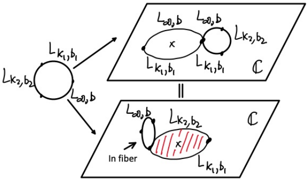


Figure: To compute ∂ (count bigons on LHS), introduce 3rd Lagrangian $\hat{\mathbf{L}}_{\infty,b}$ and use Leibniz rule (RHS - red region is ∂).

Floer theory of example

Theorem (Azam-C-Lee-Liu)

$\partial = M^1 : CF(\hat{\mathbf{L}}_{\mathsf{k}_1,z_1}, \hat{\mathbf{L}}_{\mathsf{k}_2,z_2}) \to CF(\hat{\mathbf{L}}_{\mathsf{k}_1,z_1}, \hat{\mathbf{L}}_{\mathsf{k}_2,z_2})$ computed by simplifying $M^1 M^2 = M^2(M^1(\cdot), \cdot) + M^2(\cdot, M^1(\cdot))$:

$$HF(\hat{\mathbf{L}}_{\mathsf{k}_1,z_1}, \hat{\mathbf{L}}_{\mathsf{k}_2,z_2}) \cong H^0(t^*_{z_2 - z_1} \mathcal{L}_\tau^{\otimes(\mathsf{k}_2 - \mathsf{k}_1)}|_{s_\tau^{-1}(0)})$$

where $s_\tau : (\mathbb{C}^*)^2/\tau\mathbb{Z}^2 \to \mathcal{L}_\tau$ section of degree 1 holo ample line bundle $\mathcal{L}_\tau$ and t_z: $(\mathbb{C}^*)^2/\tau\mathbb{Z}^2 \to (\mathbb{C}^*)^2/\tau\mathbb{Z}^2$ translates by z.

Key ideas: we know how to count pseudo-holomorphic triangles in a fiber and disks bounded by $\ell_{\infty,b}$ over a circle around 0 (moment map preimage), and we can keep track of areas of disks when isotoping Lagrangians to these two cases.

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