Loop Spaces and Poincaré Duality

Part I Topological Aspects

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joint work with

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M compact Riemannian manifold

(Free) Loops $\Lambda M = \{\gamma : S^1 \to M\}$

Based Loops $\Omega M = \{\gamma : (S^1, *) \to (M, *)\}$

Why loop spaces?

① Prominent role in algebraic topology

② Differential geometry

Closed geodesics on $M \equiv$ Critical points of $E : \Lambda \to \mathbb{R}$

$$E(\gamma) = \int_{S^1} |\dot{\gamma}|^2 dt$$

Morse theory : Topology of $\Lambda \leftrightarrow$ closed geodesics on M

Roughly speaking,

$H_k(\Lambda) \sim$ critical points of index k

Critical Levels

$Cr : H_k(\Lambda) \to \mathbb{R}$ $\quad Cr(X) = \inf\{a \in \mathbb{R} \mid X$ is supported on $\Lambda^{\leq a}\}$

is a critical value of E

$cr : H^k(\Lambda) \to \mathbb{R}$ $\quad cr(x) = \sup\{a \in \mathbb{R} \mid x$ is supported on $\Lambda^{\geq a}\}$

is a critical point of E

Use this correspondence to "count" closed geodesics on M.

Difficulty: If γ is a closed geodesic, then so are its iterates $\gamma^2, \gamma^3, \ldots$ $\gamma^m(t) = \gamma(mt)$

One honest geodesic $\leftrightarrow$ an infinite family of $O(2)$-orbits of critical points in Λ.

Q: Is there an algebraic structure (e.g. product) on $H_k(\Lambda)$ that corresponds to iteration of closed geodesics?

A: In many critical cases, YES.

String Topology

$\equiv$

Study of algebraic structure of $H_*(\Lambda)$.

PRODUCTS

Pontryagin product 1939

$\Omega \times \Omega \to \Omega$ concatenation

$\rightsquigarrow H_i(\Omega) \otimes H_j(\Omega) \to H_{i+j}(\Omega)$

$[A], [B] \to \left[\left\{ \alpha \cdot \beta \;\middle|\; \begin{matrix} \alpha \in A \\ \beta \in B \end{matrix} \right\}\right] =: [A \underset{PP}{\circ} B]$

Chas-Sullivan product 1999

$H_i(\Lambda) \otimes H_j(\Lambda) \to H_{i+j-n}(\Lambda)$

$[A], [B] \to \left[\left\{ \alpha \cdot \beta \;\middle|\; \begin{matrix} \alpha \in A \\ \beta \in B \\ \alpha(0) = \beta(0) \end{matrix} \right\}\right] =: [A \underset{CS}{\bullet} B]$

Coproduct $\vee_\Omega : H_k(\Omega) \to \bigoplus_{i+j=k-n+1} H_i(\Omega) \otimes H_j(\Omega)$ (Mod constant loops)

Coproduct $\vee_\Lambda : H_k(\Lambda) \to \bigoplus_{i+j=k-n+1} H_i(\Lambda) \otimes H_j(\Lambda)$ (")

product on cohomology $\circledast : H^i(\Lambda) \otimes H^j(\Lambda) \to H^{i+j+n-1}(\Lambda)$

$\langle \vee A, x \otimes y \rangle \equiv \langle A, x \circledast y \rangle$

cohomology product

Notation:

$X, Y \in H_*(\Omega)$ $\quad$ $A, B \in C_*(\Omega)$

$x, y \in H^*(\Omega)$ $\quad$ $a, b \in C^*(\Omega)$

$$X = [A]$$
$$x = [a]$$

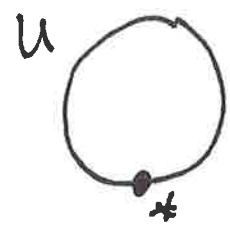
$Cr(X) \in \mathbb{R}$

$cr(x) \in \mathbb{R}$

$\alpha, \beta, \gamma \in \Lambda$

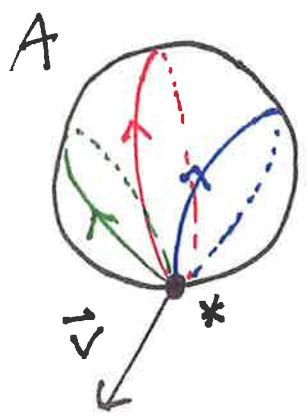
Example: $M = S^n$, $n > 1$

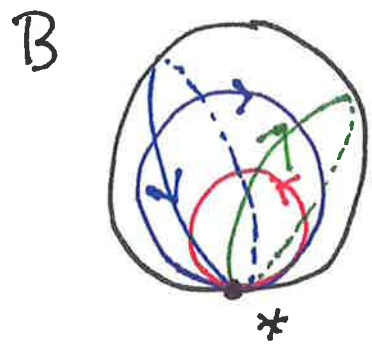
Some cycles in ΩM:



$U^0 = \{$ constant loop at $* \}$

$A^{n-1} = \left\{ \begin{array}{l} \text{Circles beginning at } * \in M \\ \text{with tangent vector } \vec{V} \end{array} \right\}$





$\vec{V}$ = tan. vector to sphere.

$B^{2n-2} = \{$ Circles beginning at $* \}$

$[u] \bullet_{pp} [A] = \underline{A}$

$[A] \bullet_{pp} [A] = \underline{B}$

$[A]$ is non nilpotent:

$[A]^m \neq 0$

in standard metric

$Cr\,[A] = \underline{2\pi}$

$Cr\,[A] \bullet_{pp} [A] = \underline{2\pi}$

$Cr\,[A]^{2m} = \underline{2m\pi}$

$Cr\,[A]^{2m-1} = \underline{2m\pi}$

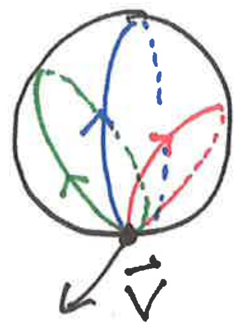
$$H_*(\Omega S^n) = \mathbb{Z}[A]$$

Pontryagin ring

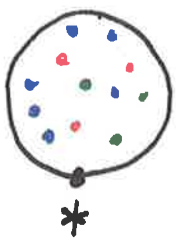
Example $M = S^n$

Some cycles in ΛM:

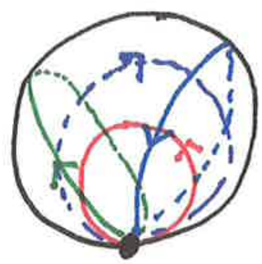
U^0 = the constant loop at $*$



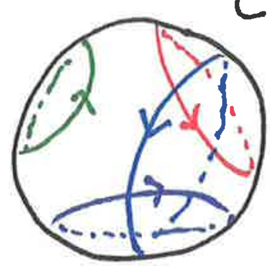
A^{n-1} = Circles beginning at $*$ with tangent vector $\vec{V}$



E^n = constant loops on S^n



B^{2n-2} = Circles beginning at $*$

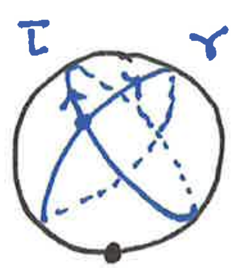


C^{3n-2} = All circles, great and small

$[C] \bullet_{cs} [E] = \underline{C}$

$[A] \bullet_{cs} [A] = \underline{0}$

$[C] \bullet_{cs} [C] = \left[\{ \gamma \cdot \tau \mid \gamma \in C, \tau \in C \text{ and } \gamma(0) = \tau(0) \} \right]$



$[C]^m \neq 0$

n even $\Rightarrow H_*(\Lambda S^n; \mathbb{Q})$ generated by U, E, A, C

(computed by Cohen-Jones-Yan.)

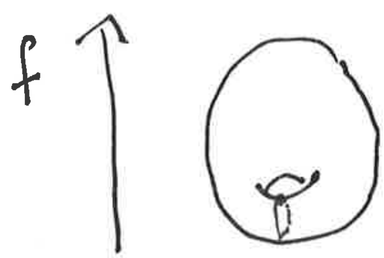
Poincaré Duality on the loop space

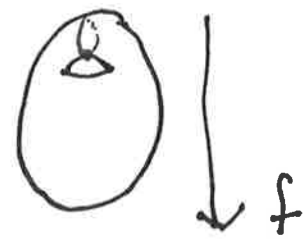
If Σ is a closed, oriented manifold of dim n,

then $PD: H^k(\Sigma) \xrightarrow{\sim} H_{n-k}(\Sigma)$.

Λ and Ω are infinite-dimensional, so the above formula does not apply. But from the point of view of Morse theory, with a Morse function f,

$$PD: (\Sigma, f) \leftrightarrow (\Sigma, -f)$$





Morse k-chain on Σ

Morse cochain of dim $n-k$

Poincaré duality on the loop space says,

Given M, Λ, Ω, E, many things "look the same" if you turn the energy function E upside-down.

Powerful Principle

Many examples

Hard to find a good statement

Evidence for Poincaré Duality

(1) Morse theory inequalities

$$X, Y \in H_*(\Lambda) \qquad x, y \in H^*(\Lambda, \Lambda^0)$$

$\Longrightarrow$ $$\mathrm{Cr}(X \underset{CS}{\bullet} Y) \leq \mathrm{Cr}(X) + \mathrm{Cr}(Y)$$

Goresky-H 2009

$$\mathrm{cr}(x \circledast y) \geq \mathrm{cr}(x) + \mathrm{cr}(y)$$

Note: One inequality involves homology, the other cohomology. One involves the CS product, the other the cohomology product. The inequalities have different signs.

<u>Cor</u> $$\mathrm{Cr}(mX) \leq m\,\mathrm{Cr}(X)$$

$$\mathrm{cr}(mx) \geq m\,\mathrm{cr}(x)$$

Evidence for PD

(2) Growth of the index of the iterates of a closed geodesic

$$m \cdot \mathrm{Index}(\gamma) - (m-1)(n-1) \leq \mathrm{Index}(\gamma^m) \leq m \cdot \mathrm{Index}(\gamma) + (m-1)(n-1)$$

Bott 1956

Equality → ~~Fastest~~ possible.

$$m\,\mathrm{index}(\gamma) - (m-1)(n-1) \leq \mathrm{Index}(\gamma^m) \leq m\,\mathrm{Index}(\gamma) + (m-1)(n-1)$$

Slowest possible growth

Evidence for PD

3) Suppose all closed geodesics on M are nondegenerate. Given $X \in H_*(\Lambda)$, $x \in H^*(\Lambda, \Lambda^0)$ $\exists$ $m \in \mathbb{N}$ so that

$$Cr(X^m) < m\, Cr(X)$$

$$cr(x^m) > m\, cr(x).$$

4) If $\exists X \in H_*(\Lambda)$ with $Cr(X) > 0$ and
(H 93) $Cr(X^m) = m\, Cr(X)$ for all m, then M has infinitely many closed geodesics.

If $\exists$ $x \in H^*(\Lambda, \Lambda^0)$ with $x \neq 0$ and
(H 97) $cr(x^m) = m\, cr(x)$ for all m, then M has infinitely many closed geodesics.

Evidence for PD

5) Gysin Formulas (Goresky - H 2009)

$$i : \Omega \to \Lambda \rightsquigarrow i_! : H_*(\Lambda) \to H_{*-n}(\Omega)$$

$$i^! : H^*(\Omega, *) \to H^{*+n}(\Lambda, \Lambda^\circ)$$

Let $A, B \in H_*(\Lambda)$, $C \in H_*(\Omega)$

$a, b \in H^*(\Lambda, \Lambda^\circ)$, $c \in H^*(\Omega, *)$.

Then

$$\begin{cases} i_!(A \underset{\uparrow \atop cs}{\cdot} B) = i_!(A) \underset{\uparrow \atop pp}{\bullet} i_!(B) \\ i_*(C) \underset{\uparrow \atop cs}{\bullet} A = i_*(C \underset{\uparrow \atop pp}{\bullet} i_! A) \end{cases}$$

$$\begin{cases} i^*(a \circledast b) = i^*(a) \circledast i^*(b) \\ i^!(c) \circledast a = i^!(c \circledast i^*(a) \end{cases}$$

(6) New Definitions $\wedge, \vee$

$e = \mathrm{eval} : \Lambda \to M$

$e \times e : \Lambda \times \Lambda \to M \times M$

U_ε nbhd of Δ in $M \times M$

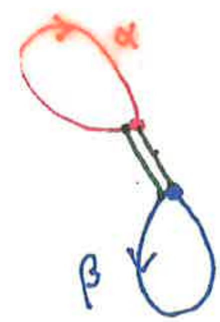
$e \times I : \Lambda \times I \to M \times M$

$(x,t) \mapsto (x(0), x(t))$

$$A \wedge B =: \mathrm{concat}\left(R\left([e \times e]^*(\tau_M) \cap A \times B\right)\right)$$

$$\vee A =: \mathrm{cut}\left(R\left([e \times I]^*(\tau_M) \cap A \times I\right)\right)$$

$$R_{\Lambda \times \Lambda} : \underbrace{(e \times e)^{-1}(U_\varepsilon)}_{\text{nbhd of } F_{\Lambda \times \Lambda} \text{ in } \Lambda \times \Lambda} \longrightarrow F_{\Lambda \times \Lambda}$$



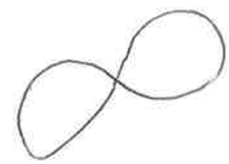
$$R_{\Lambda \times I} : \underbrace{(e \times I)^{-1}(U_\varepsilon)}_{\text{nbhd of } F_{\Lambda \times I} \text{ in } \Lambda \times I} \longrightarrow F_{\Lambda \times I}$$

Theorem: these chain maps represents the CS product and coproduct.

modulo boundary

... small chains ...

Figure-8 spaces



$$F_{\Lambda\times\Lambda} = (e\times e)^{-1}(\Delta M) \subset \Lambda\times\Lambda$$
$$= \{(\alpha,\beta) \mid \alpha(0)=\beta(0)\}$$

$$F_{\Lambda\times I} = (e\times I)^{-1}(\Delta M) \subset \Lambda\times I$$
$$= \{(\alpha,t) \mid \alpha(0)=\alpha(t)\}$$

Idea of CS product: $A, B \in C_*(\Lambda)$

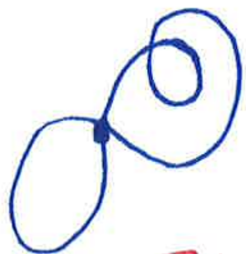
$$A\times B \to (A\times B)\cap F \xrightarrow[\text{concetenate}]{} A\cdot B$$

Idea of coproduct: $A \in C_*(\Lambda)$

$$A \longrightarrow A\times I \longrightarrow (A\times I)\cap F \xrightarrow[\text{cut}]{} \Lambda A$$

What do the products on ΛM tell us about M?

<u>Coproduct</u>: "Look for intersections and cut"
$(\alpha(0) = \alpha(t))$



Should be true:

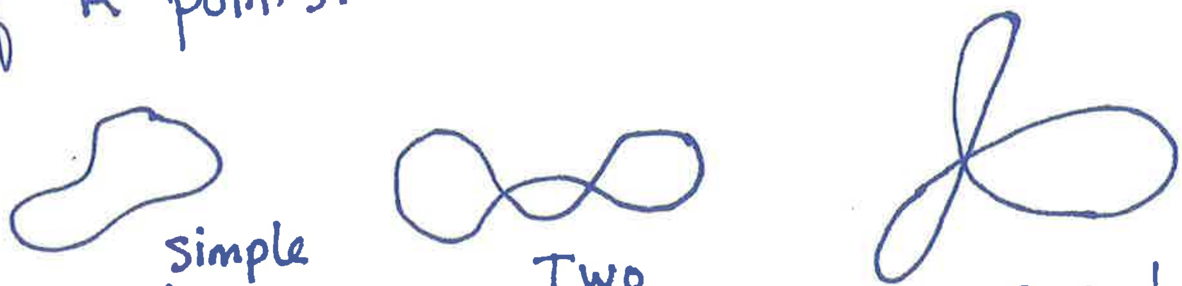
If A is supported on simple loops, then

$[\vee A] = 0$ (in homology).

This should be the most basic property of the coproduct. We were unable to prove it using the old definitions!

<u>But</u> with the new definitions we were able to do even <u>better</u>:

Def: A loop $\gamma \in \Lambda$ has a <u>k-fold intersection</u> at $p \in M$ if $\gamma^{-1}\{p\}$ consists of k points.



simple loop. (1-fold intersections)

Two 2-fold intersections

a 3-fold intersection.

Theorem If $Z \subset \Lambda$ is a cycle & every nonconstant loop in the image of Z has at most k-fold intersections, then

$$\hat{V}^k [Z] = 0.$$

<u>Sharp</u> for spheres, projective spaces!

Sharp for spheres and projective spaces:

Given $X \in H_*(\Lambda)$, we define the intersection number

$$\operatorname{int}(X) = \inf_{\substack{A \in C_*\Lambda \\ [A]=X}} \ \sup_{\substack{\ell(\gamma)>0 \\ p\in M}} \#(\gamma^{-1}\{p\}) \quad \#(\gamma^{-1}\{p\})$$

Theorem (H-Wahl)

Let M be a sphere or a projective space, $X \in H_*\Lambda$, and $k \geq 1$. Then

$$\operatorname{int}(X) \leq k \iff \hat{\vee}^{k}(X) = 0.$$

(any coefficients)

Moral: For spheres and projective spaces, the coproduct is doing just what it should do: It predicts self-intersections.

The proof uses:

(1) $\exists$ a metric with all geodesics closed, and of the same period ℓ

(2) $H_*(\Lambda^{\leq \ell})$ supported on the union of the simple and the constant loops.

(3) Products!

Lifted Coproduct

$$A \in C_*(\Lambda) \Rightarrow \vee A \in C_*(\Lambda \times \Lambda, \underbrace{\Lambda \times \Lambda^0 \cup \Lambda^0 \times \Lambda}_{\text{Boundary terms}})$$

<u>Theorem</u> (H-Wahl)

$\exists !$ lift $\hat{\vee} : C_*(\Lambda) \to C_*(\Lambda \times \Lambda)$ chain map

satisfying $\langle x \times y, \hat{\vee} C \rangle = 0$

if $x \in e^* C^*(M)$, $y \in e^* C^*(M)$ or $Z \in C_*(M)$