2023.01.20. MSRI Eigenvariety for partially classical Hilbert modular forms



## §1. Intro and Result

$Gl_2/\mathbb{Q}$ $\quad$ $\{\boxed{\text{classical}}^{(I=\Sigma)}$ modular forms$\}$ $\subseteq$ $\{\boxed{\text{overconvergent}}^{(I=\phi)}$ p-adic MFs$\}$

$Gl_2/F$ $\quad$ Hilbert $\quad\cap\quad\subseteq\quad$ Hilbert

$F$: totally real field $\qquad \{I\text{-classical HMFs}\}$

$[F:\mathbb{Q}]=g$

$\Sigma = \{\infty_1,\dots,\infty_g : F \hookrightarrow \mathbb{R} \subseteq \mathbb{C}\}$

$\cup\!|$ $\qquad\qquad\qquad\qquad\qquad \mathrel{S\!|\!|}$

$I$ $\qquad\qquad\qquad\qquad\qquad\quad \overline{\mathbb{Q}}_p$

Look at Hecke eigenvalues of these spaces

(Coleman–Mazur '98) (Andreatta–Iovita–Pilloni '16)

$\exists$ eigenvar. $\mathcal{E}$: $g$

$\exists$ eigencurve $\mathcal{E}$ : 1-dim'l rigid analytic space $/\mathbb{Q}_p$

$\downarrow$

weight space $\mathcal{W}$

$\mathcal{E}$ parametrizes finite slope Hecke e.v.s of OCMFs (OCHMFs)

(i.e. $U_p$-e.v. $\neq 0$)

$\mathcal{W} = (\mathrm{Spf}\ \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!])^{rig}$ $\quad$ $((\mathcal{O}_F\otimes\mathbb{Z}_p)^\times)$ $\quad$ param. analytic char.s $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ $\quad ((\mathcal{O}_F\otimes\mathbb{Z}_p)^\times)$

eg. $z \mapsto z^k$ is weight $k\in\mathbb{Z}$

**Theorem** (in progress)

Assume $p$: unram.

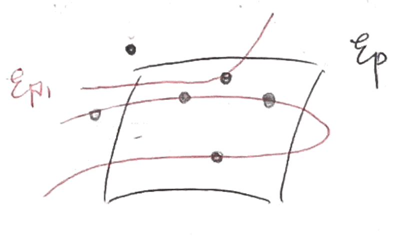
Let $t \mid p$, $\Sigma_t := \{\infty : F\hookrightarrow \overline{\mathbb{Q}}_p \text{ inducing } \mathfrak{p}\mid t\}$ $\quad$ ($t \subset \mathcal{O}_F$)

(Let $\underline{k}\in \mathbb{Z}_{\geq 2}^{\Sigma_{\frac{p}{t}}}$)

Then $\exists$ eigenvar. $\mathcal{E}_t^{(\underline{k})}$ param. "finite $\frac{p}{t}$-slope" Hecke e.v.s of $\Sigma_t$-classical HMFs

(dim $= g-\#\Sigma_t$) $\downarrow$ $\qquad$ ($U_{\mathfrak{p}}$-e.v. $\neq 0$ $\ \forall\, \mathfrak{p}\mid\frac{p}{t}$)

$\left(\mathcal{W}_t = \mathrm{Spf}\ \mathbb{Z}_p[\![\prod_{\mathfrak{p}\mid t}\mathcal{O}_{F_\mathfrak{p}}^\times]\!])^{rig}\right)$




$p=p_1p_2$

**Application**

Let $f\in\mathcal{E}_t \rightsquigarrow \rho_f : \mathrm{Gal}_F \to Gl_2(\overline{\mathbb{Q}}_p)$

Then $\rho_f|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is de Rham $\forall\, \mathfrak{p}\mid t$

## §2. AIP's idea – the MF case

$$\begin{array}{l} E^{univ} \\ \downarrow \\ X : \text{cpt modular curve, level prime to } p \end{array}$$

$\omega = \pi_* \Omega^1_{E^{univ}/X}$

$\{\text{MFs of wt } k \in \mathbb{Z}\} = H^0(X, \omega^k)$

Want: modular sheaves $\omega^\kappa$ for analytic $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$

Idea: torsor construction of $\omega^k$

$$\begin{array}{l} T = \mathrm{Isom}(\mathcal{O}_X, \omega) \\ \quad \downarrow \ \mathbb{G}_m\text{-torsor} \\ \quad X \end{array}$$

$$\Rightarrow \underset{\substack{\circlearrowleft \\ \mathbb{G}_m}}{(\pi_* \mathcal{O}_T)}\underline{[-k]} = \omega^k \qquad \text{acts as } \begin{array}{c} \mathbb{G}_m \to \mathbb{G}_m \\ z \mapsto z^{-k} \end{array}$$

$\mathfrak{X}$ : completion of $X$ along $\bar{X}/\mathbb{F}_p$

$\mathcal{X}$ : rigid generic fiber of $\mathfrak{X}$

The Hodge height is $\mathrm{Hdg} : \mathcal{X} \to [0,1] \cap \mathbb{Q}$
$$x \mapsto \inf(\mathrm{val}_p(\widetilde{\mathrm{Ha}}(x)), 1).$$

$v \in [0,1] \cap \mathbb{Q} \rightsquigarrow \mathcal{X}(v) = \{x \in \mathcal{X} : \mathrm{Hdg}(x) \le v\}$ is a strict nbd. of $\mathcal{X}(0)$ = ordinary locus of $\mathcal{X}$

$\{\text{OCMF of wt } k \in \mathbb{Z}\} = \varinjlim_{v \to 0^+} H^0(\mathcal{X}(v), \omega^k)$


Suppose $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ is $n$-analytic

$$\begin{array}{ccc} \mathbb{Z}_p^\times & \to & \mathbb{C}_p^\times \\ \cup & & \cup \\ 1 + p^n\mathbb{Z}_p & & 1 + p\mathcal{O}_{\mathbb{C}_p} \\ \log \downarrow & & \uparrow \exp \\ p^n\mathbb{Z}_p & \xrightarrow[\psi]{} & p\mathcal{O}_{\mathbb{C}_p} \end{array}$$

Pick $v < \frac{1}{p^n}$. Then over $\mathcal{X}(v)$, $\exists$ canonical subgp $\underset{\substack{\text{SII in char. 0} \\ \mathbb{Z}/p^n\mathbb{Z}}}{H_n} \subseteq E[p^n]$ ($H_1 \bmod p^{1-v} = \ker \mathrm{Frob}$).

$$\pi \left( \begin{array}{l} T_w = \mathrm{Isom}(\mathcal{O}, \mathcal{F}) \\ \downarrow \ 1 + p^w\mathbb{Z}_p\text{-torsor} \\ \mathcal{X}(v, p^n) = \mathrm{Isom}(\mathbb{Z}/p^n\mathbb{Z}, H_n) \\ \downarrow \ (\mathbb{Z}/p^n\mathbb{Z})^\times\text{-torsor} \\ \mathcal{X}(v) \end{array} \right.$$

(compatible with $HT_w$ and $\mathbb{Z}/p^n\mathbb{Z} \cong H_n$)

On $\mathfrak{X}(v, p^n)$, $\exists \left(p^{\frac{v}{p-1}}\omega \subseteq\right) \mathcal{F} \subseteq \omega$ s.t.

the Hodge-Tate map $HT : \begin{array}{c} H_n^D \to \omega_{H_n} \to \omega_E \\ H_n \xrightarrow[h]{} \mathbb{G}_m \mapsto h^* \frac{dt}{t} \end{array}$ induces Isom mod $p^w$

$HT_w : H_n^D / p^w \to \mathcal{F}/p^w$ for $w \le n - v \cdot \frac{p^n}{p-1}$

**Def.** $\kappa$ : n-ana., $v < \frac{1}{p^n}$, $w \leq n - v \cdot \frac{p^n}{p-1}$

Define $\omega_w^{\kappa} = (\pi_* \mathcal{O}_{\mathcal{T}_w})[-\kappa]$

**Rmk** For $n-1 \leq w \leq n - v \cdot \frac{p^n}{p-1}$, all $\omega_w^{\kappa}$ are canonically isom. $\rightsquigarrow$ denote by $\omega_n^{\kappa}$

For $n' \geq n$, $\omega_n^{\kappa}$ and $\omega_{n'}^{\kappa}$ are isom.

It turns out $\omega_w^{\kappa}$ is indep. of $w, n \rightsquigarrow \omega^{\kappa}$

## §3. Partially classical HMFs

$X/\mathbb{Z}_p$ : Hilbert modular variety

$\mathcal{O}_F \curvearrowright \omega = \pi_* \Omega^1_{A^{univ}/X} \rightsquigarrow \omega = \bigoplus_{i=1}^{g} \omega_i$ $\quad \mathcal{O}_F \curvearrowright$ on $\omega_i$ via $\sigma_i$

For $\underline{k} = (k_i) \in \mathbb{Z}^g$, $\omega^{\underline{k}} = \otimes \omega_i^{k_i}$

Have partial Hasse inv.s $Ha_i \rightsquigarrow Hdg_i$

For $\underline{v} \in ([0,1] \cap \mathbb{Q})^g$, $X(\underline{v}) = \{x \in X, Hdg_i(x) \leq v_i\}$

For $I \subseteq \Sigma$,

$$\left\{ \begin{array}{l} I\text{-classical HMF} \\ \text{of wt } \underline{k} \in \mathbb{Z}^g \end{array} \right\} = \lim_{\substack{v_i \to 0+,\ i \notin I \\ v_i = 1,\ i \in I}} H^0(X(v_i), \omega^{\underline{k}})$$

Key for constructing $\mathcal{E}_I$ is the existence of partial canonical subgp.

When $I = \Sigma_t$, this can be done

$$\sup_{i \in \Sigma_p} v_i < \frac{1}{p^n} \Rightarrow \exists\ H_{p,n} \subseteq A[p^n], \quad H_{p,1} \bmod p^{1-\sup v_i} = \ker \mathrm{Frob}_p$$

## §4. Application

**Thm** Assume the existence of $\mathcal{E}_I$

Let $f$ be a $I$-classical HMF, Hecke eigen

$\rightsquigarrow \rho_f : \mathrm{Gal}_F \to GL_2(E)$, $E \supseteq \mathbb{Q}_p$

Then $\rho_f$ is $I$-de Rham (eg. when $I = \Sigma_t$, $\rho_f|_{\mathrm{Gal}_{F_p}}$ is de Rham $\forall p | t$)

<u>Proof idea</u>

(1) Partial classicality thm ( $I$-classical HMF of small slope is classical )

$\Rightarrow$ classical MFs form a dense set of $\mathcal{E}_I$

$\Rightarrow$ $\rho$ is de Rham on this dense set

(2) For $i \in I$, the $\infty_i$ - Hodge - Tate weight of $\rho$ does not change.

<u>Thm ( Berger - Colmez )</u>

In a Galois deform space where the HT wt.s are bdd.

the de Rham locus is a closed subspace

$\Rightarrow$ For $i \in I$, $\rho$ is $\infty_i$ - de Rham

<u>Def</u> ( partial deRham )

- $\rho_f|_{Gal_{F_p}}$ is de Rham if $\dim_{F_p} \underset{\substack{\| \\ (V_p \otimes_{\mathbb{Q}_p} B_{dR})^{Gal_{F_p}}}}{D_{dR}(V_p)} = \dim_{\mathbb{Q}_p} V_p = 2[E:\mathbb{Q}_p]$

$$\underset{\substack{\uparrow \\ F_p\text{-v.s}}}{B_{dR}} \otimes_{\mathbb{Q}_p} E = \bigoplus_{i=1}^{g} B_{dR,i} \ , \quad D_{dR}(V_p)_i = (V_p \otimes_E B_{dR,i})^{Gal_{F_p}}$$

$$( F_p \otimes_{\mathbb{Q}_p} E = \bigoplus_{i=1}^{g} E )$$

- $\rho_f|_{Gal_{F_p}}$ is $\infty_i$-de Rham if $\dim_{F_p} D_{dR}(V_p)_i = \dim_E V_p = 2$

- $\rho_f$ is $I$-de Rham if it is $\infty_i$ - de Rham $\forall\, \infty_i \in I$.