# Supersingular Loci of Unitary Shimura Varieties

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# Objectives

1. Discuss structure of “low-dimensional” examples of supersingular loci
2. Introduce main tool for studying supersingular loci
3. See how the general situation differs from the “low-dimensional” examples

# Motivation: Modular Curves over $\mathbb{C}$

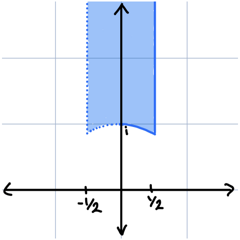
Seek to understand

$$\mathcal{Y}(\mathbb{C}) = \{\text{elliptic curves } E \text{ over } \mathbb{C}\}/\cong$$

- Define
$$\begin{aligned} \mathcal{H} &\to \mathcal{Y}(\mathbb{C}) \\ \tau &\mapsto \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \end{aligned}$$

- $\mathbb{C}/(\mathbb{Z} + \tau_1\mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z} + \tau_2\mathbb{Z})$ if and only if $\tau_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau_1$.

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \cong \mathcal{Y}(\mathbb{C})$$




- Easy to understand $\mathcal{Y}(\mathbb{C})$ as a complex manifold, since elliptic curves over $\mathbb{C}$ are "linear algebraic."

# Motivation: Modular Curves over $\overline{\mathbb{F}}_p$

Seek to understand

$$\mathcal{Y}(\overline{\mathbb{F}}_p) = \{\text{elliptic curves } E \text{ over } \overline{\mathbb{F}}_p\}/\cong$$

## $j$-invariant (Dedekind or Klein, 1800's)

There is a bijection:

$$\begin{array}{ccc} \mathcal{Y}(\overline{\mathbb{F}}_p) & \leftrightarrow & \overline{\mathbb{F}}_p \\ E : y^2 = x^3 + Ax + B & \mapsto & j(E) = \frac{1728(4A)^3}{16(4A^3+27B^2)} \\ E_j : y^2 = x^3 - \frac{1}{4}x^2 - \frac{36}{j-1728}x - \frac{1}{j-1728} & \leftarrow\!\shortmid & j \end{array}$$

Note: this construction is not "linear-algebraic"!

# Motivation: Modular Curves over $\overline{\mathbb{F}}_p$

- An elliptic curve $E/\overline{\mathbb{F}}_p$ can be ordinary or supersingular.
- The supersingular locus $\mathcal{Y}(\overline{\mathbb{F}}_p)^{ss} \subseteq \mathcal{Y}(\overline{\mathbb{F}}_p)$ parametrizes supersingular elliptic curves.

**Cor. to Eichler-Deuring Mass Formula:**

There are approx. $\frac{p}{12}$ supersingular elliptic curves over $\overline{\mathbb{F}}_p$.

# $\mathrm{GU}(a, b)$ Shimura Variety

Fix a quad. im. field $K$ and $p \neq 2$ inert in $K$

**The $\mathrm{GU}(a, b)$ Shimura variety $\mathcal{M}(a, b)$**

parametrizes $(A, \iota, \lambda, \eta)$:

- $A$ an A.V. of dim a+b
- $\iota$ an action of $\mathcal{O} \subseteq K$

Meeting the signature $(a, b)$ condition:

$$\det(T - \iota(k); \mathrm{Lie}(A)) = (T - \varphi_1(k))^a (T - \varphi_2(k))^b.$$

Example: Let $E : y^2 = x^3 - x$. $\mathbb{Z}[i]$ acts on $E$, where:

$$\mathbf{i} : E(\mathbb{C}) \to E(\mathbb{C})$$
$$(x, y) \mapsto (-x, iy)$$

Define $\mathbb{Z}[i]$-action on $A = E \times E \times E$ as:

# GU$(a, b)$ Shimura Variety

## The GU$(a, b)$ Shimura variety $\mathcal{M}(a, b)$

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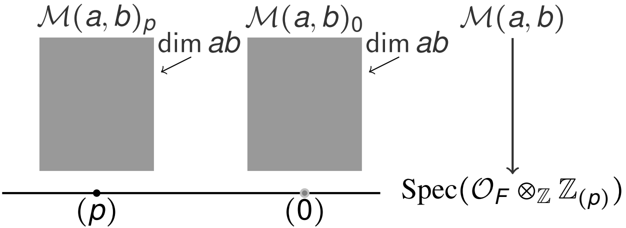


$\mathcal{M}(a, b)_0(\mathbb{C}) \cong \bigsqcup_{i=1}^{n} \Gamma_i \backslash \mathcal{H}_{a,b}$, analogous to $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$.

# $\mathrm{GU}(a,b)$ Shimura Variety

## The $\mathrm{GU}(a,b)$ Shimura variety $\mathcal{M}(a,b)$

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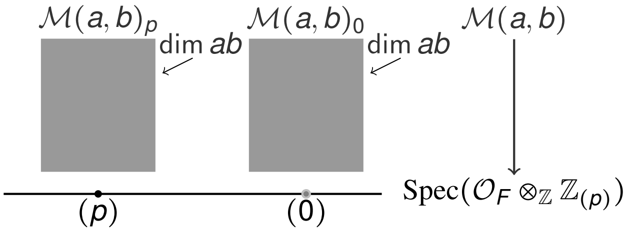



The supersingular locus $\mathcal{M}(a,b)_p^{ss}$ parametrizes $(A,\iota,\lambda,\eta)$ where $A$ is supersingular.

# Results on Geometry of $\mathcal{M}(a,b)_p^{ss}$

- The geometry of $\mathcal{M}(a,b)_p^{ss}$ depends on the signature $(a,b)$. $\mathcal{M}(a,b) \cong \mathcal{M}(b,a)$, so take $b \geq a \geq 0$.
- The supersingular loci $\mathcal{M}(a,b)_p^{ss}$ have been described by...

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $(0,m)$ | $(1,1)$ | $(1,2)$ | $(1,m-1)$, $m \geq 4$ | $(2,2)$ | $(2,3)$, $(2,4)$ | $(2,m-2)$, $m \geq 7$ | $(a,m-a)$ $a \geq 3$ $m \geq 6$ |
| 0-dim'l | 0-dim'l | Vollaard 2008 | Vollaard-Wedhorn 2010 | Howard-Pappas 2014 | Imai-F. 2021 after perf. | Imai-F. 2021 (partial) | Incomplete |

- For motivation, we'll consider the $(1,2) = (2,1)$ and $(2,2)$ cases.
- We'll see how the $(2,4)$ case both generalizes the $(1,2)$ and $(2,2)$ cases, and contains new structure.

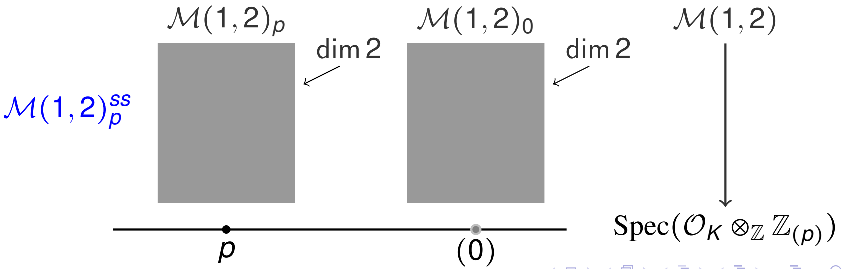
# Ex: The Supersingular Locus of $\mathcal{M}(1,2)$

## Theorem (Vollaard '08)

Assume $\eta$ is suff. small. Each irreducible component of $\mathcal{M}(1,2)_p^{ss}$ is isomorphic to the Fermat curve

$$C : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$

There are $p^3 + 1$ int. pts on each irr. comp., and each int. point is the intersection of $p+1$ irr. comps.



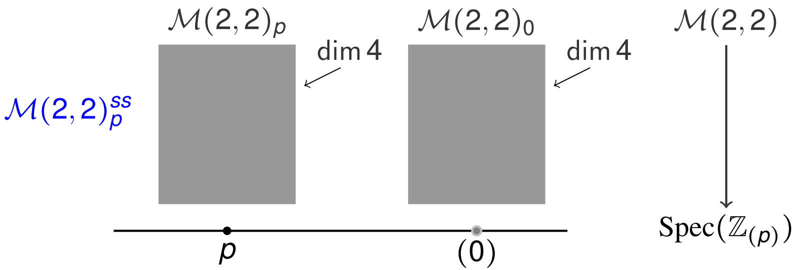
# Ex: The Supersingular Locus of $\mathcal{M}(2,2)$

**Theorem (Howard-Pappas '14)**

Assume $\eta$ is suff. small. Each irreducible component of $\mathcal{M}(2,2)_p^{ss}$ is isomorphic to the Fermat surface

$$S : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} + x_3^{p+1} \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p}.$$

Any two irr. components intersect trivially, in a projective line, or in a point.



# Deligne-Lusztig Varieties

Moduli spaces of flags in char. $p$ vector spaces, with "fixed relative position" to Frobenius-twist:

Given $G$ over $\mathbb{F}_p$, $B$, and $w \in N_G(T)/T$:

$$X(w) = \{gB \in G/B \mid g^{-1}Fr(g) \in BwB\}.$$

Example: $G = \mathrm{SL}_2$, $B$ upper-tri, $N_G(T)/T = \{1, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\}$.

- $G/B \cong \{$ lines $\ell \subseteq \overline{\mathbb{F}}_p^2\}$
- $\mathrm{rel}(\ell_1, \ell_2) = 1$ if and only if $\ell_1 = \ell_2$
- If $\ell = \langle c_o e_o + c_1 e_1 \rangle$, $\mathrm{Fr}(\ell) = \langle c_o^p e_o + c_1^p e_1 \rangle$
- So,

$$X(1) = \{\ell \subseteq \overline{\mathbb{F}}_p^2 \mid \mathrm{rel}(\ell, \mathrm{Fr}(\ell)) = 1\} = \mathbb{P}^1(\mathbb{F}_p)$$

$$X\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \{\ell \subseteq \overline{\mathbb{F}}_p^2 \mid \mathrm{rel}(\ell, \mathrm{Fr}(\ell)) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\} = \mathbb{P}^1(\overline{\mathbb{F}}_p) \setminus \mathbb{P}^1(\mathbb{F}_p).$$

# General Situation

- The signature $(1,2)$ and signature $(2,2)$ supersingular loci have similar structure: their irr. components are Deligne-Lusztig varieties, intersection combinatorics come from a Bruhat-Tits building, the supersingular locus is a union of Ekedahl-Oort Strata.
- These are examples of Coxeter Type. Görtz, He, and Nie have classified which supersingular loci have this structure, after perfection. First paper (2013) lists all 21 possibilities.
- Most unitary Shimura varieties do not have this structure.

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# Strategy: Rapoport-Zink Uniformization

- If $A$ is an abelian variety of dim. $m$ over $\mathbb{C}$, study lattice $\Lambda \subseteq \mathbb{C}^m$ such that:
$$A(\mathbb{C}) \cong \mathbb{C}^m/\Lambda.$$
($\mathbb{Z}$-module of rank $2m$)
- If $A$ is an abelian variety of dim. $m$ over $L$, and $p \nmid \mathrm{char}(L)$, study:
$$T_p(A) = \varprojlim A[p^k](\overline{L})$$
($\mathbb{Z}_p$-module of rank $2m$)
- If $A$ is a supersingular abelian variety of dim. $m$ over $\overline{\mathbb{F}}_p$, $T_p(A) = 0$. Instead, study the p-divisible group:
$$A[p^\infty] = \varinjlim A[p^k].$$
Equivalently, study the p-adic Dieudonné module $D_p(A)$ ($\breve{\mathbb{Z}}_p$-module of rank $2m$, with operator $F$)

# Unitary Rapoport-Zink Spaces

Unitary Rapoport-Zink Space: $\mathcal{N}(a,b)(S) = \{(G,\iota,\lambda,\rho)\}/\cong$,

- $G$ a supersingular $p$-div. gp over $S$ of dim $a+b$
- $\iota : \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(G)$ of sign. $(a,b)$
- $\rho : G_{S_0} \to \mathbb{G}_{S_0}$, quasi-isog

## Rapoport-Zink Uniformization

$$\mathcal{M}(a,b)_p^{ss} \cong \bigsqcup_{j=1}^{m} \Gamma_j \backslash \mathcal{N}(a,b)$$

The $\Gamma_j$ are discrete groups (depending on level structure) acting on $\mathcal{N}(a,b)$.

Can study the (more "linear-algebraic") Rapoport-Zink spaces $\mathcal{N}(a,b)$ to understand the supersingular loci $\mathcal{M}(a,b)_p^{ss}$

# Unitary Rapoport-Zink Spaces

Unitary Rapoport-Zink Space: $\mathcal{N}(a,b)(\overline{\mathbb{F}}_p) = \{(G,\iota,\lambda,\rho)\}/\cong,$

$$\mathcal{N}(a,b)(\overline{\mathbb{F}}_p) = \{M \subseteq \mathbb{N} \mid pM \subseteq FM \subseteq M,\ \mathcal{O}-\text{stable},\ M = p^i M^\vee\}$$

## Rapoport-Zink Uniformization

$$\mathcal{M}(a,b)_p^{ss} \cong \bigsqcup_{j=1}^{m} \Gamma_j \backslash \mathcal{N}(a,b)$$

The $\Gamma_j$ are discrete groups (depending on level structure) acting on $\mathcal{N}(a,b)$.

Can study the (more "linear-algebraic") Rapoport-Zink spaces $\mathcal{N}(a,b)$ to understand the supersingular loci $\mathcal{M}(a,b)_p^{ss}$

# The $\mathrm{GU}(1,2)$ Rapoport-Zink Space

## Thm (Vollaard): Geometry of $\mathcal{N}(1,2)$

- Each irr. comp. of $\mathcal{N}(1,2)$ is isom to:

$$C \ : \ x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0 \subset \mathbb{P}^2_{\overline{\mathbb{F}}_p}.$$

  (a Deligne-Lusztig variety). Any two irr. components intersect trivially or in a single point.
- Each irr. comp. contains $p^3 + 1$ int. pts, and each int pt is the intersection of $p + 1$ irr. comp.
- $\mathcal{N}(1,2)$ has two Ekedahl-Oort strata: the intersection points, and their complement.

# Why Expect Deligne-Lusztig Varieties?

- Replace $(G, \iota, \lambda, \rho)$ with p-adic Dieudonné module $M$ to identify:

$$\mathcal{N}(1,2)(\overline{\mathbb{F}}_p) = \{M \subseteq \mathbb{N} \mid \text{ conditions wrt } F\}.$$

- (Convert from $\mathbb{N}$ to an alternative Hermitian space $W$: )

$$\mathcal{N}(1,2)(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid \text{ conditions wrt } F\}.$$

- Irreducible components $\mathcal{N}_\Lambda \subset \mathcal{N}(1,2)$ defined as:

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \{L \subseteq W \mid p\Lambda \subseteq L \subseteq \Lambda \text{ \& conditions wrt } F\}.$$

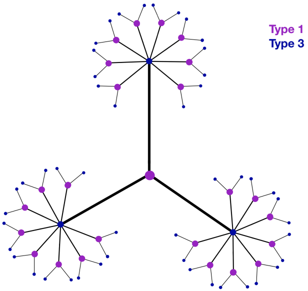
- Replace $L$ with $\ell = L/p\Lambda$

$$\mathcal{N}_\Lambda(\overline{\mathbb{F}}_p) = \{\ell \subseteq (\Lambda/p\Lambda)_{\overline{\mathbb{F}}_p} \mid \text{conditions wrt } F\},$$

a Deligne-Lusztig variety.

# Bruhat-Tits Building

- From the relevant hermitian space $W$ for $\mathcal{N}(1,2)$, can construct a Bruhat Tits building $\mathcal{B}$:
- $\mathcal{B}$ is an (infinite) tree, with two types of vertices:
  "Type-1" ($\Lambda \subseteq W$ s.t. $p\Lambda^\vee \underset{1}{\subseteq} \Lambda \subseteq \Lambda^\vee$ )
  "Type-3" ($\Lambda \subseteq W$ s.t. $p\Lambda^\vee \underset{3}{\subseteq} \Lambda \subseteq \Lambda^\vee$ )
  Type-1 vertices have degree $p+1$, Type-3 have $p^3+1$.



# Ekedahl-Oort Strata

- $(G_1, \iota_1, \lambda, \rho_1)$ and $(G_2, \iota_2, \lambda, \rho_2)$ are in the same Ekedahl-Oort stratum of $\mathcal{N}(1,2)$ if and only if $(G_1[p], \iota_1, \lambda_1) \cong (G_2[p], \iota_2, \lambda_2)$.

- $\mathcal{N}(1,2)$ has two Ekedahl-Oort strata: the intersection points, and their complement.

- The intersection points are exactly those points where $G[p] \cong E[p]^3$ for a supersingular elliptic curve $E$.

# The $\mathrm{GU}(1,2)$ Rapoport-Zink Space Again

## Thm (Vollaard): Geometry of $\mathcal{N}(1,2)$

- Each irr. comp. of $\mathcal{N}(1,2)$ is isom to:
$$C \;:\; x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0 \subset \mathbb{P}^2_{\mathbb{F}_p}.$$
(a Deligne-Lusztig variety). Any two irr. components intersect trivially or in a single point.
- Each irr. comp. contains $p^3 + 1$ int. pts, and each int pt is the intersection of $p + 1$ irr. comp.
- $\mathcal{N}(1,2)$ has two Ekedahl-Oort strata: the intersection points, and their complement.

# The $\mathrm{GU}(1,2)$ Rapoport-Zink Space Again

**Thm (Vollaard): Geometry of $\mathcal{N}(1,2)$ (Rephrased)**

- Each irr. comp. of $\mathcal{N}(1,2)$ is isom to a particular kind of Deligne-Lusztig variety
- The intersection combinatorics come from a relevant Bruhat-Tits building
- $\mathcal{N}(1,2)$ is a union of Ekedahl-Oort strata

# The $\mathrm{GU}(2,2)$ Rapoport-Zink Space

**Thm (Howard-Pappas):**

- Each irr. comp. of $\mathcal{N}(2,2)$ is isom to:
$$S : x_0^{p+1} + x_1^{p+1} + x_2^{p+1} + x_3^{p+1} = 0 \subset \mathbb{P}^3_{\overline{\mathbb{F}}_p}.$$
Any two irr. comp. intersect trivially, intersect in a single point, or have intersection isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$.
- For a fixed irreducible component $X$, there are $p(p^3+1)(p^2+1)$ irreducible components $X'$ such that $X \cap X'$ is a single point and $(p^3+1)(p+1)$ irreducible components $X''$ such that $X \cap X''$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$.

# General Situation

- The signature $(1,2)$ and signature $(2,2)$ Rapoport-Zink spaces have similar structure: their irr. components are Deligne-Lusztig varieties, intersection combinatorics come from a Bruhat-Tits building, are a union of Ekedahl-Oort Strata.
- These are examples of Coxeter Type.
- Most unitary Rapoport-Zink spaces do not have this structure.

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# The $\mathrm{GU}(2, m-2)$ Rapoport-Zink Space

**Thm (Imai-F.):**

Let $m \geq 5$:

- $\mathcal{N}(2, m-2)^{\mathrm{perf}}$ is $(m-2)$-dimensional, and contains $\lfloor \frac{m}{2} \rfloor$ isomorphism classes of irreducible components.
- One component $X$ generalizes those of in $\mathcal{N}(1, 2)$, and is a Deligne-Lusztig variety
- If $m$ is even, there is a component $Y$ generalizing those of $\mathcal{N}(2, 2)$, and is a Deligne-Lusztig variety.
- The remaining irreducible components are not Deligne-Lusztig varieties. We describe them via a map to a Deligne-Lusztig variety.
- When $m = 5, 6$, describe all intersections of irr. comps.

# Comments

- This theorem crucially uses the results in "Cycles on Shimura Varieties..." by Xiao and Zhu. This project arose from discussions at the 2019 AIM workshop "Geometric Realizations of Jacquet-Langlands Correspondences."

- Joint with Naoki Imai and Ben Howard, currently studying the structure of $\mathcal{N}(2, m-2)$ (as opposed to $\mathcal{N}(2, m-2)^{\mathrm{perf}}$)

- Beginning at the 2022 Rethinking Number Theory Workshop, currently studying properties of the Ekedahl-Oort stratification (joint with D. Bhamidipati, H. Goodson, S. Groen, S. Nair, E. Stacy).

# The $\mathrm{GU}(2,4)$ Rapoport-Zink Space

Thm (Imai-F.):

- $\mathcal{N}(2,4)^{\mathrm{perf}}$ contains three isomorphism classes of irreducible components: $X$, $Y$, and $Z$. We describe these as closures in a certain flag scheme
- The component $X$ generalizes those occurring in $\mathcal{N}(1,2)$. The component $Y$ generalizes those occurring in $\mathcal{N}(2,2)$. Both $X$ and $Y$ are Deligne-Lusztig varieties.
- The final irreducible component $Z$ is notable in that it is not isomorphic to a Deligne-Lusztig variety.
- We describe all possible intersections of irr. components.

# Notation

- Let $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{nr}}$, with ring of integers $\breve{\mathbb{Z}}_p$.
- $V$ is an $m$-dimensional vector space over $\breve{\mathbb{Q}}_p$, with $\breve{\mathbb{Q}}_p$-valued Hermitian form.
- For any lattice $L \subseteq V$, $L^\vee$ denotes the dual lattice.
- $\Lambda_0 \subseteq V$ a fixed self-dual $\breve{\mathbb{Z}}_p$-lattice.

# Relative Position of Lattices

- Given lattices $L_1$ and $L_2$ in $V$, we say $\mathrm{Inv}_{L_1}(L_2) = (n_1, n_2, \ldots, n_m)$ if $L_1 = \mathrm{Span}_{\breve{\mathbb{Z}}_p}\{e_i\}_{i=1}^m$ and $L_2 = \mathrm{Span}_{\breve{\mathbb{Z}}_p}\{p^{n_i} e_i\}_{i=1}^m$.

- Example: $L_1 = \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$, $L_2 = p^2\breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$. Then $\mathrm{Inv}_{L_1}(L_2) = (2, 0, 0)$.

- Example: $L_1 = \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$, $L_2 = p\breve{\mathbb{Z}}_p \oplus p\breve{\mathbb{Z}}_p \oplus \breve{\mathbb{Z}}_p$. Then $\mathrm{Inv}_{L_1}(L_2) = (1, 1, 0)$.

# A Component Generalizing Vollaard-Wedhorn

Define $X \subseteq \mathcal{N}(2,4)^{\mathrm{perf}}$ as:

$$\begin{array}{ccc} X & \longrightarrow & \mathring{\mathrm{Gr}}^{0}_{(\nu_1^*, \mu^*)|\nu_1} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1 \times \mathrm{m}} \\ \mathring{\mathrm{Gr}}_{\nu_1^*} & \xrightarrow{1\times\sigma} & \mathring{\mathrm{Gr}}_{\nu_1^*} \times \mathring{\mathrm{Gr}}_{\nu_1} \end{array}$$

$$\nu_1 = (1,0,0,0,0,0)$$
$$\nu_1^* = (0,0,0,0,0,-1)$$
$$\mu^* = (0,0,0,0,-1,-1).$$

Meaning:

$$X(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq V \;\middle|\; \begin{array}{l} \mathrm{Inv}_{\Lambda_0}(L) = \nu_1^*, \\ \mathrm{Inv}_L(F(L^\vee)) = \mu^*, \\ \mathrm{Inv}_{\Lambda_0}(F(L^\vee)) = \nu_1 \end{array} \right\}$$

# A Component Generalizing Vollaard-Wedhorn

- Rephrase as a single chain condition:

$$X(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq V \;\middle|\; \begin{array}{l} \mathrm{Inv}_{\Lambda_0}(L) = (1,0,0,0,0,0), \\ \mathrm{Inv}_L(F(L^\vee)) = (0,0,0,0,-1,-1), \\ \mathrm{Inv}_{\Lambda_0}(F(L^\vee)) = (0,0,0,0,0,-1) \end{array} \right\}$$

$$= \left\{ \text{Lattices } L \subseteq V \;\middle|\; p\Lambda_0 \underset{1}{\subsetneq} pF(L^\vee) \underset{1}{\subsetneq} L \underset{1}{\subsetneq} \Lambda_0 \right\}$$

- Realized as a Deligne-Lusztig variety as:

$$\left\{ L \subseteq V \mid p\Lambda_0 \underset{1}{\subsetneq} pF(L^\vee) \underset{1}{\subsetneq} L \underset{1}{\subsetneq} \Lambda_0 \right\} \xrightarrow{\sim} \{\ell \subseteq \Lambda_0/p\Lambda_0 \mid \dim(\ell) = 1,\ \ell \subseteq \ell^\perp\}$$

$$L \mapsto pF(L^\vee)/p\Lambda_0$$

- If $V$ were dim 3 instead, this recovers the Fermat curve appearing in $\mathcal{N}(1,2)$.

# A Component Generalizing Howard-Pappas

- Similarly, define $Y \subseteq \mathcal{N}(2,4)^{\mathrm{perf}}$:

$$Y(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq V \;\middle|\; \begin{array}{l} \mathrm{Inv}_{\Lambda_0}(L) = (0,0,0,0,-1,-1), \\ \mathrm{Inv}_L(\frac{1}{p}F(L^\vee)) = (0,0,0,0,-1,-1), \\ \mathrm{Inv}_{\Lambda_0}(\frac{1}{p}F(L^\vee)) = (0,0,-1,-1,-1,-1) \end{array} \right\}$$

- $Y$ is isomorphic to a Deligne-Lusztig variety
- If $V$ were dim 4 instead, this recovers the Fermat surface appearing in $\mathcal{N}(2,2)$.

# A New Component

- There is an open subset $\mathring{Z}$ of $Z$ defined by:

$$\mathring{Z}(\overline{\mathbb{F}}_p) = \left\{ \text{Lattices } L \subseteq V \;\middle|\; \begin{array}{l} \mathrm{Inv}_{\Lambda_0}(L) = \nu_2^*, \\ \mathrm{Inv}_L(F(L^\vee)) = \mu^*, \\ \mathrm{Inv}_{\Lambda_0}(F(L^\vee)) = \nu_2 \end{array} \right\}^{a}$$

$$\nu_2 = (1,0,0,0,-1,-1) \quad \nu_2^* = (1,1,0,0,0,-1)$$

$$\mu^* = (0,0,0,0,-1,-1).$$

- $\mathring{Z}$ is not a Deligne-Lusztig variety, but does map to one:

$$\pi : \mathring{Z} \to \mathrm{Gr}_{\nu_{2,+}^*} \times \mathrm{Gr}_{\nu_{2,-}^*}$$

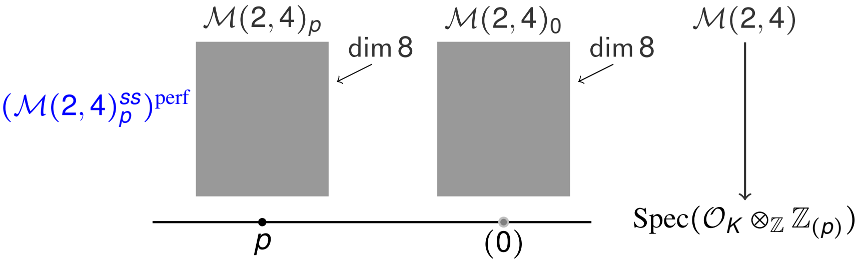
$$L \mapsto (L \cap \Lambda_0, L + \Lambda_0)$$

- $Z \setminus \mathring{Z} \cong \{[x_0 : x_1 : \cdots : x_5] \in \mathbb{P}^5_{\overline{\mathbb{F}}_p} \mid \sum_{i=0}^5 x_i^{p+1} = 0,\ \sum_{i=0}^5 x_i^{p^3+1}\}$

## The Supersingular Locus of $\mathcal{M}(2,4)$

### Cor (Imai-F.):

Assume $\eta$ is suff. small. $(\mathcal{M}(2,4)_p^{ss})^{\mathrm{perf}}$ contains three isomorphism classes of irreducible components: $X$, $Y$, and $Z$, which we describe concretely. The component $Y$ is notable in that it is not isomorphic to a Deligne-Lusztig variety.
We describe all possible intersections of irr. components.



# Some Takeaways

1. The Rapoport-Zink spaces $\mathcal{N}(a, b)$ occur naturally in the study of $\mathcal{M}(a, b)_p^{ss}$, and are closely related to the supersingular loci $\mathcal{M}(a, b)_p^{ss}$.
2. In some cases, the supersingular loci $\mathcal{M}(a, b)_p^{ss}$ have especially nice structure (can be written as a union of Deligne-Lusztig varieties, intersection combinatorics controlled by a B-T building, are a union of Ekedahl-Oort strata.)
3. Warning! Not all Shimura varieties of PEL-type (or even all unitary Shimura varieties) have this nice structure.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $(0, m)$ | $(1, 1)$ | $(1, 2)$ | $(1, m-1)$, $m \geq 4$ | $(2, 2)$ | $(2, 3)$, $(2, 4)$ | $(2, m-2)$, $m \geq 7$ | $(a, m-a)$ $a \geq 3$ $m \geq 6$ |
| 0-dim'l | 0-dim'l | Vollaard 2008 | Vollaard-Wedhorn 2010 | Howard-Pappas 2014 | Imai-F. 2021 after perf. | Imai-F. 2021 (partial) | Incomplete |

Thank you!