Green's functions for local systems over graphs

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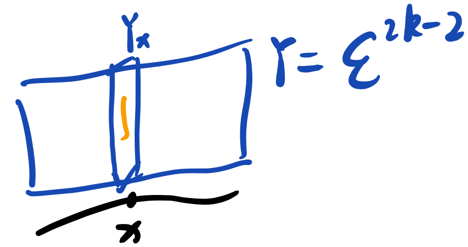
Outline:

1. Number theoretic context.
2. Constructions of Green's functions.

* The board talk differs from the slides here, but this is a sketch of the talk.

Number theoretic Context.



The Gross – Zagier – Zhang Formula:

Height pairings of Heegner divisors on modular curves

Heegner cycles in Kuga-Sato varieties over modular curves

Derivatives of certain L-series associated to

with reference to an imaginary quadratic field K

Certain weight-2 modular forms.

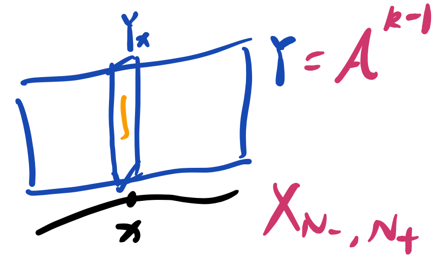
2k

of level $\Gamma_0(N)$

primes dividing N splits in K

Number theoretic Context.

? Generalization of



The Gross – Zagier – Zhang Formula :

Height pairings of Heegner divisors on modular curves

Heegner cycles in Kuga-Sato varieties over modular curves

CM cycles in Kuga varieties over some Shimura curves

Derivatives of certain L-series associated to

Certain weight-2 modular forms. with reference to an imaginary quadratic field K

$2k$

of level $\Gamma_0(N)$

$N = N_- N_+$

N_- inert in K ; N_+ split in K

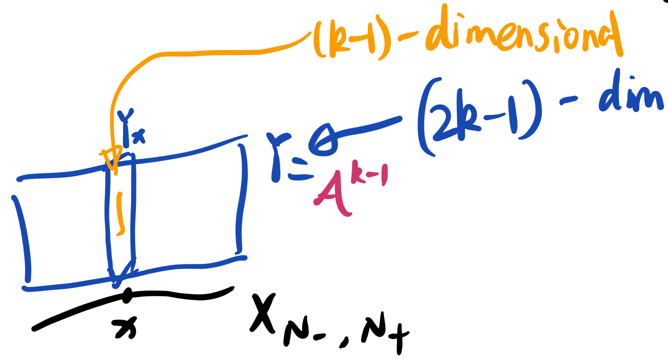
N_-: product of an even number of distinct primes

$\boxed{k = 1 \text{ Case covered in [Zhang 2001] and [Yuan-Zhang-Zhang 2013]}}$

This generalization is work in progress with Congling Qiu and Zhiyu Zhang. In this talk I'll focus on the progress made in [W. 2022].

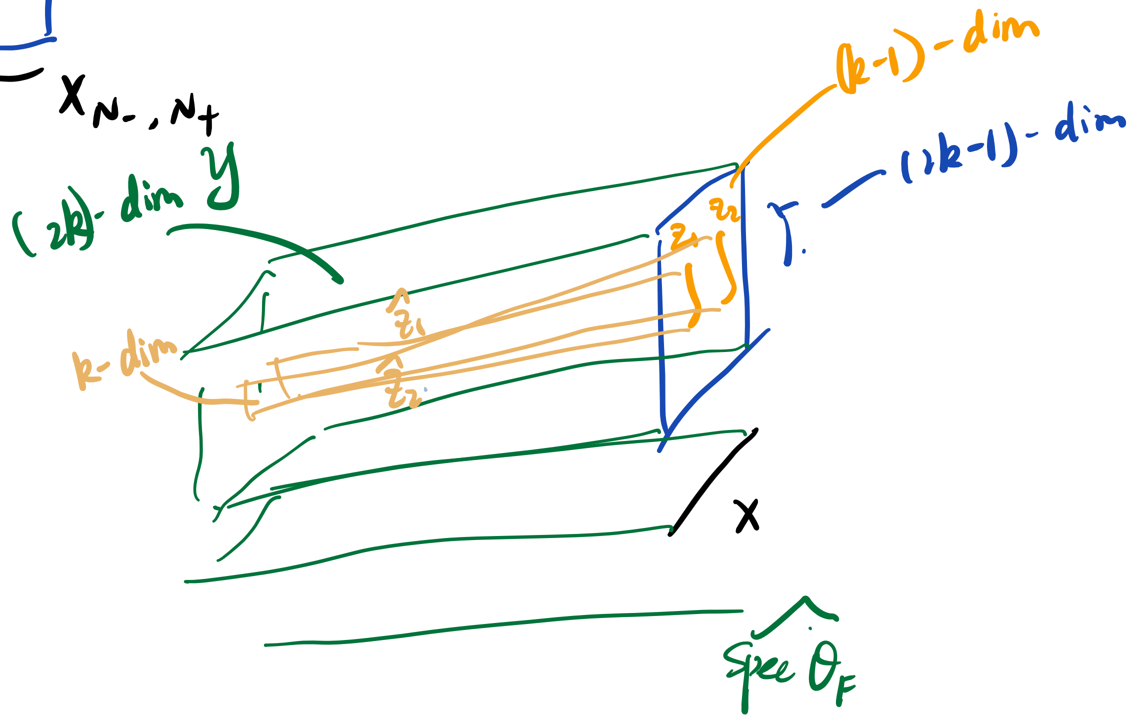
First, a bit more about the height pairing:

The cycles:



Gillet - Soulé intersection pairing

(Beilinson - Bloch height pairing)



First need to find suitable arithmetic extensions $\hat{z}_1$, $\hat{z}_2$.

In particular, for any finite place $\wp$ and any cycle z of appropriate dimension supported on $\mathcal{Y}_\wp$,

$$(\hat{z}_1, z)_\wp = 0 \quad (*)$$

Unfortunately, for the cycles we are interested in, $\bar{z}_1$ does not satisfy this. Need to find adjustments $z_\wp$ s.t.

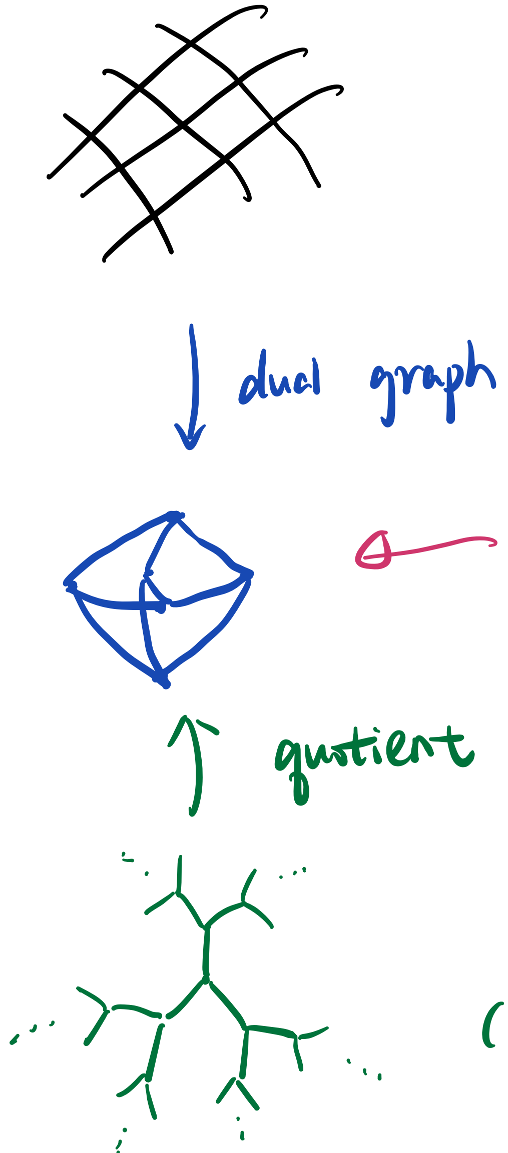
$$\hat{z}_1 := \bar{z}_1 - \sum_{\wp | N_-} z_\wp \quad \text{satisfies } *$$

Finding $z_\wp$ becomes solving $\Delta f = C\delta_w$ on local systems on graphs.

To solve such an equation, we need to construct Green's functions for local systems over graphs

What kind of graphs?

By Cerednik-Drinfeld, X_{N_-,N_+} at $p \mid N_-$:



we are interested in such graphs. Finite quotients of the Bruhat-Tits tree

Bruhat-Tits tree
(homogeneous $(p+1)$-deg tree)

Constructions of Green's functions for local systems over graphs

Also studied before, see e.g. [Jordan & Livné 1997]

Let $G = (V, E)$ be a finite graph.

set of vertices

set of edges

A local system $\mathcal{L}$ over G of rank r contains the data:

for each $v \in V$, $\mathcal{L}(v)$
for each $e \in E$, $\mathcal{L}(e)$
} Hermitian spaces of dim r

AND

whenever v is an end of e,
$\varphi_{ve} : \mathcal{L}(v) \to \mathcal{L}(e)$, an isometric isomorphism

Define

sections above vertices

$$\Delta : \Gamma(V, \mathcal{L}) \to \Gamma(V, \mathcal{L}) \quad \text{by}$$

$$\Delta f(v) = \sum_{\substack{e \text{ has} \\ v, v' \text{ as} \\ \text{ends}}} f(v) - \varphi_{ve}^{-1} \varphi_{v'e} f(v')$$

We have

$$\Gamma(V, \mathcal{L}) = \ker(\Delta) \oplus \operatorname{Im}(\Delta)$$

orthogonal

So for any $\phi \in \ker(\Delta)^{\perp} = \operatorname{Im}(\Delta)$, there exists a unique $f \in \operatorname{Im}(\Delta)$ s.t. $\Delta f = \phi$

<u>Def</u>. The vector-valued Green's fcn G associated with $\mathcal{L}$ on a graph $\mathcal{G} = (V, E)$ is defined to be the unique elt $G \in \mathrm{Im}(\Delta) \otimes \mathrm{Im}(\Delta)$ s.t.

$$\Delta_v G(v, w) = \delta_w(v) \quad \text{as distributions.}$$

Therefore, given the Green's fcn $G = \sum_i a_i \otimes b_i$, for any $\varphi \in \mathrm{Im}(\Delta)$, the unique solution in $\mathrm{Im}(\Delta)$ to $\Delta f = \varphi$ is given by

$$\begin{aligned} f(w) &= \sum_{v \in V} \langle \varphi(v), G(v, w) \rangle \\ &= \sum_{v \in V} \sum_i \langle \varphi(v), a_i(v) \rangle \overline{b_i(w)} \end{aligned}$$

We also define G_s, replacing "Δ" above by "$\Delta + s$".

Thm (W.) We construct $G(v,w)$ explicitly for finite graphs that are quotients of the Bruhat-Tits tree

$$\mathcal{G} = \Gamma \backslash \mathcal{T}$$

Method:

- Construct $G_s(v,w)$ for $s>0$
- $G(v,w) = \lim_{s \to 0^+} \Big(\underbrace{G_s(v,w)}_{\sum_{\gamma \in \Gamma} g_s(\gamma \tilde{v}, \tilde{w})} - \frac{\chi}{s} \Big)$

g_s on $\mathcal{T}$