Cycles, motives and Langlands

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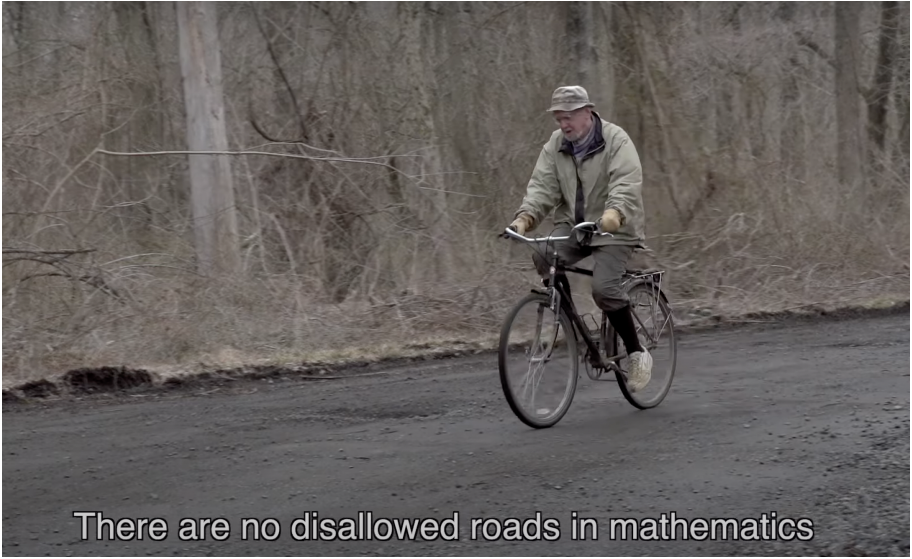
Overview

1. Cycles and Langlands
2. Elliptic curves over $\mathbf{Q}$
3. The Hodge and Tate conjectures
4. Elliptic curves over real quadratic fields
5. A theorem and sketch of the proof
6. Other recent related work and a preview of next lecture

Cycles and Langlands

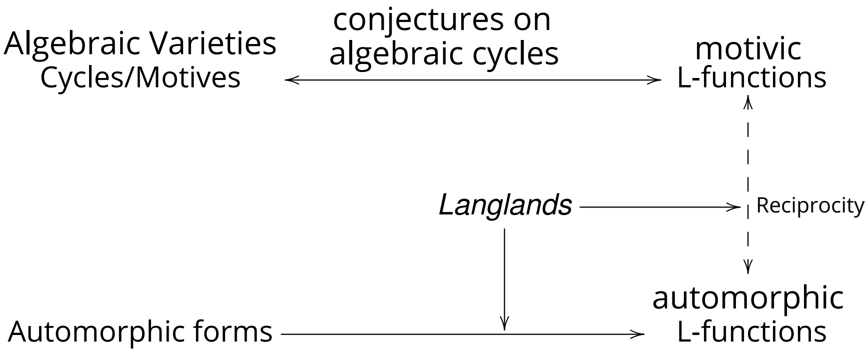


Cycles and Langlands (contd.)

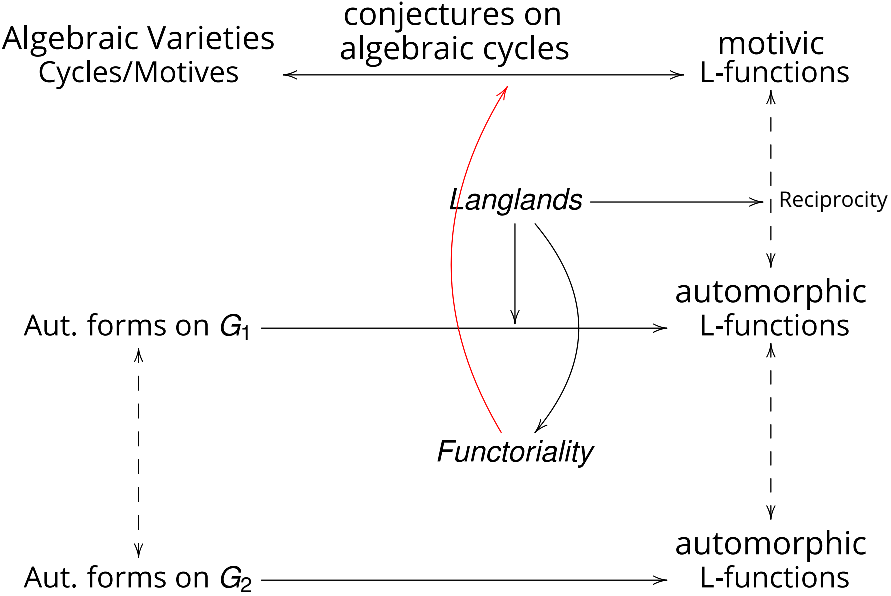
- Some open problems about algebraic cycles:
  1. Birch and Swinnerton-Dyer conjecture (relates rational points on elliptic curves or abelian varieties to the behavior of the L-function at the central point $s = 1$)
  2. Bloch-Beilinson conjecture (relates ranks of Chow groups of varieties to the behavior of the L-function at the central point)
  3. Beilinson's conjectures (studies motivic L-functions at integer points away from the center; relates them to higher Chow groups)
  4. Bloch-Kato conjecture (refined version of the above relating the exact value to arithmetic invariants; a vast generalization of the class number formula)
  5. Hodge conjecture and the Tate conjecture (related to the standard conjectures on algebraic cycles)
- The formulation of these conjectures has nothing to do with the Langlands program.
- But the hope is that Langlands can help us study these questions!

Cycles and Langlands (contd.)

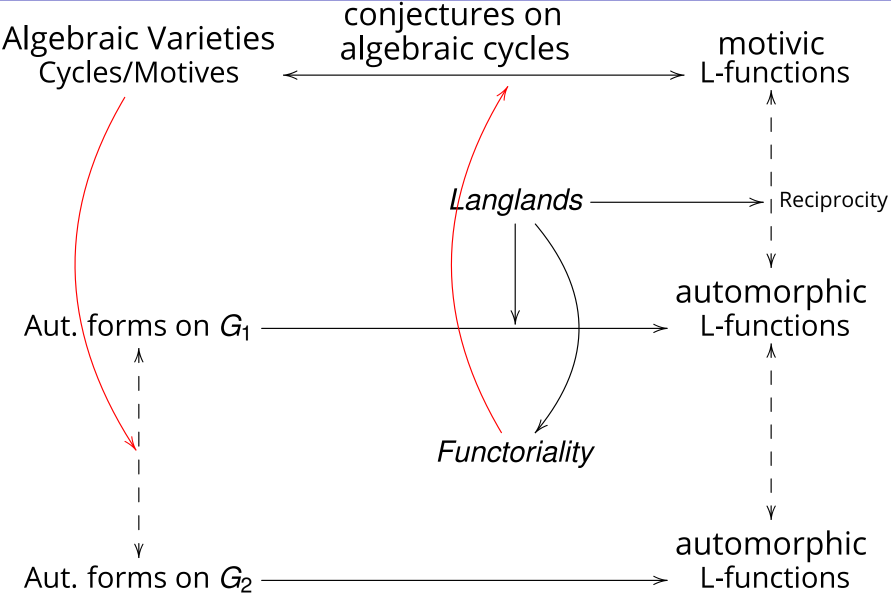
Why do we expect Langlands to play a role?



Cycles and Langlands (contd.)



Cycles and Langlands (contd.)



A few related things I won't talk about

- The Kudla program: arithmetic intersection theory/ Arakelov theory
  Related to the theory of Borcherds' products
  Applications: Colmez's conjecture relating Faltings heights of CM abelian varieties to logarithmic derivatives of Artin L-functions
  What is the relation with the picture on the previous page?
- Cycles on varieties over finite fields.
  eg. the work of Liang Xiao/ Xinwen Zhu and related developments
- Function field case
  eg. moduli spaces of shtukas (Tony Feng's talks)

Elliptic curves over **Q**

- The basic example:
  Let E be an elliptic curve over $\mathbf{Q}$:

$$y^2 = x^3 + ax + b$$

  The BSD conjecture relates the group of rational points $E(\mathbf{Q})$ to an analytic object, the L-function of E.

$$L(E, s) = \prod_p L_p(E, s)$$

  where for almost all p,

$$L_p(E, s) = (1 - a_p(E)p^{-s} + p^{1-2s})^{-1}$$

$$1 - a_p(E) + p = \#E(\mathbb{F}_p)$$

- (Conjecture) rank $E(\mathbf{Q}) = \mathrm{ord}_{s=1} L(E, s)$

Elliptic curves over $\mathbf{Q}$, contd.

Some of the main ideas to understand this:

- E is modular: this means (among other things) that there is a surjective map
$$X_0(N) \to E$$
for some modular curve $X_0(N)$.
- (Gross-Zagier theorem) One considers *special cycles* on $X_0(N)$, and then map them down to E.

Let's unpack the modularity part a bit:

- Modularity: there exists a modular form f on $\Gamma_0(N)$, which is an eigenform for the Hecke operators, such that the pth Hecke eigenvalue of f equals $a_p(E)$ for almost all p.
- We can think of this in terms of cohomology: for $X = X_0(N)$, the Hecke algebra acts on
$$H^1_{sing}(X(\mathbf{C}), \mathbf{Q}), \quad H^1_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$$

Elliptic curves over $\mathbf{Q}$, contd.

- Let J be the Jacobian variety of X. The Eichler-Shimura relation tells us that

$$H^1_{et}(J_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^1_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^1(E_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$$

as $G_{\mathbf{Q}}$-modules. $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ associated with f, and the subscript π denotes the π or f-isotypic component, i.e., the part on which the Hecke algebra acts by the same Hecke eigenvalues as those of f.)

- Faltings (Mordell/Tate conjecture, 1983) $\implies$ there exists a nonconstant map

$$J \to E$$

which induces the isomorphism above. Composing it with some map $X \to J$, we get a nonconstant map $X \to E$.

Hodge and Tate conjectures

- Let X be a smooth projective variety over $\mathbf{C}$.
- Cycles of codimension j: $Z^j(X)$ is the free abelian group on codimension j subvarieties. This is equipped with an equivalence relation called rational equivalence. The Chow group
$$\mathrm{CH}^j(X) = Z^j(X)/\sim$$
- There is a cycle class map
$$\mathrm{cl} : \mathrm{CH}^j(X) \otimes \mathbf{Q} \to H^{2j}_{sing}(X(\mathbf{C}), \mathbf{C})$$
- The RHS carries two important structures. One is a $\mathbf{Q}$-structure, namely
$$H^{2j}_{sing}(X(\mathbf{C}), \mathbf{Q})$$
- The other is a Hodge structure:
$$H^{2j} = \oplus_{p+q=2j} H^{p,q}$$

Hodge and Tate conjectures, contd.

- The image of cl lies in the intersection

$$H^{2j}_{sing}(X(\mathbf{C}), \mathbf{Q}) \cap H^{j,j}$$

  This is called the space of *Hodge classes*.

- The Hodge conjecture is the statement that cl is surjective onto the space of Hodge classes. In other words, every Hodge class is represented by an algebraic cycle with **Q**-coefficients.
- The Tate conjecture is a similar statement for ℓ-adic cohomology. In this case, we start with a variety X over a number field K. Then there is a cycle class map

$$\mathrm{cl}_\ell : \mathrm{CH}^j(X) \otimes \mathbf{Q}_\ell \to H^{2j}_{et}(X_{\overline{K}}, \mathbf{Q}_\ell)$$

- The RHS carries an action of G_K.

Hodge and Tate conjectures, contd.

- The image of cl_ℓ lies in the G_K-invariants:

$$H^{2j}_{et}(X_{\overline{K}}, \mathbf{Q}_\ell)^{G_K}$$

  These classes are called *Tate classes*.
- The Tate conjecture is the statement that cl_ℓ is surjective onto the space of Tate classes.
- If X is defined over a number field K, and we fix an embedding $K \hookrightarrow \mathbf{C}$, we may consider X as a complex variety. Then a cycle on X gives rise to a Hodge class, and Tate classes for all ℓ.

$$\mathrm{comp}_\ell : H^{2j}_{sing}(X_\sigma(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{Q}_\ell \simeq H^{2j}_{et}(X_{\overline{K}}, \mathbf{Q}_\ell)$$

  In other words, we get in this way Hodge classes that are Galois invariant for all ℓ.

Hodge and Tate conjectures, contd.

- How does Faltings fit into this? Modularity + Eichler-Shimura gives

$$H^1_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^1(E_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$$

which may be viewed as giving a Tate class on the product

$$X \times E$$

- If the Tate conjecture holds in this case, then this class is represented by an algebraic cycle on $X \times E$.
- This cycle induces a map

$$\mathrm{CH}^1(X)_0 \to \mathrm{CH}^1(E)_0$$

which is exactly the map $J \to E$.

Elliptic curves over real quadratic fields

- Let F be real quadratic and E an elliptic curve over F.
- Modularity is known: there exists a Hilbert modular form f associated with E.
- We try to imitate the construction over $\mathbf{Q}$. But this runs into a problem. The most natural analog of $X_0(N)$ is a Hilbert modular surface X, attached to the algebraic group $GL_{2,F}$.
- But $H^1(X) = 0$, so there are no Tate cycles on $X \times E$.
- **Functoriality** comes to the rescue!
- Idea: look at quaternion algebras B over F, and the algebraic group $B^\times$. Jacquet and Langlands give very precise conditions for the form f to *transfer* to $B^\times$. (This means there is an automorphic form on $B^\times$ with the same Hecke eigenvalues at all but finitely many places.)

Elliptic curves over real quadratic fields, contd.

- If we pick such a B_1 satisfying further

$$B_1 \otimes_{\sigma_1} \mathbf{R} = M_2(\mathbf{R})$$

  but

$$B_1 \otimes_{\sigma_2} \mathbf{R} = \mathbb{H}$$

  then the associated "Shimura variety" is a curve X_1, Moreover, $H^1(X_1)_\pi$ is nonzero, and we get a Tate cycle on $X_1 \times E$.
- So we get a map

$$J_1 \to E$$

  which we can use to construct the analog of Heegner points etc.
- Similarly we can use a B_2 which is split at the second infinite place but ramified at the first. This would give a Tate cycle on $X_2 \times E^\sigma$ where σ is the Galois involution of $F/\mathbf{Q}$.

Elliptic curves over real quadratic fields, contd.

- What happens if we vary the quaternion algebras?
- Let's use the following notation:
  B, B' etc. for quaternion algebras that are split at both infinite places σ_1, σ_2. X, X' etc. for the corresponding Shimura varieties. These are two dimensional.
  B_1, B_1' etc. : split at σ_1, but ramified at σ_2; X_1, X_1' curves.
  B_2, B_2' etc. split at σ_2, but ramified at σ_1; X_2, X_2' curves.
- The key is to understand the Galois representations on the cohomology (Sug Woo Shin's talks)
- The system of Hecke eigenvalues associated to E occurs in $H^1(X_1)$ and $H^1(X_2)$ but in $H^2(X)$. Moreover, as $G_{\mathbf{Q}}$-modules:

$$H^2_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^1_{et}(X_{1,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \otimes_i H^1_{et}(X_{2,\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi$$

Elliptic curves over real quadratic fields, contd.

- In particular, as $G_{\mathbf{Q}}$-modules:

$$H^2_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^2_{et}(X'_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi$$

- This gives a Tate class ξ_ℓ in $H^4(X \times X')$. Question: is this represented by an algebraic cycle?
- This is (already) an extremely difficult case of the Tate conjecture.
- Weaker version of this question: can we find a Hodge class ξ that interpolates between all the ξ_ℓ? (Deligne: absolute Hodge classes).

A theorem

- **Theorem** (Ichino-P.) There exists a Hodge class ξ on $X \times X'$ such that
$$\xi_\ell := \mathrm{comp}_\ell(\xi)$$
is a Tate class for all ℓ. Moreover ξ and ξ_ℓ respectively induce isomorphisms of rational Hodge structures :
$$H^2_{sing}(X(\mathbf{C}), \mathbf{Q})_\pi \simeq H^2_{sing}(X'(\mathbf{C}), \mathbf{Q})_\pi$$
and of $G_{\mathbf{Q}}$-representations:
$$H^2_{et}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi \simeq H^2_{et}(X'_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)_\pi$$
- ArXiv preprint from 2018, currently being revised.

Some key ideas

Basic Idea: make a map

$$X \times X' \xrightarrow{\iota} Y$$

where Y is some auxiliary (Shimura) variety, find a Hodge class η on Y and pull back to $X \times X'$.
Steps:

1. Guess Y!
2. Compute the cohomology of Y.
3. Look for Hodge classes on Y. (Good situation: $H^{2p}(Y)_\Pi = H^{p,p}(Y)_\Pi$ for some representation Π.)
4. Find a Hodge class η on Y and show that $\xi := \iota^*\eta$ works. This last step usually involves some computation of periods:

$$\int_{X\times X'} \iota^*(\eta) \cdot \omega_\pi \neq 0 \quad ??$$

The construction

- Think of X and X' as unitary Shimura varieties, corresponding to unitary spaces V and V'. The embedding

$$U(V) \times U(V') \to U(W)$$

where $W = V \oplus V'$ gives rise to a map $X \times X' \to Y$.

- Identify a suitable automorphic representation Π on $U(W)$. (Use classification of automorphic forms into local and global A-packets)
Adams-Johnson, Aranciaba-Moeglin-Renard
Arthur, Mok, Kaletha-Minguez-Shin-White
- Prove the nonvanishing of the period.
The last item is very tricky!

Periods

- General problem

$$\begin{array}{ccc} G & \Pi & \eta \\ \Big| & & \\ H & \pi & \omega \end{array}$$

  Study the period

$$\int_{[H]} \eta|_H \cdot \omega$$

  Is it non-zero for some suitable choices of η, ω ?
- Representation-theoretic problem: does the restriction of Π to H contain π as a (sub)quotient?
- Gan-Gross-Prasad (GGP) conjecture
- Relative Langlands program (Sakellaridis-Venkatesh)

Periods (contd.)

- What do we do? The situation we're interested in doesn't fall into any of these categories (as yet).
- Trick: give an explicit construction of Π that allows the computation of the period.
- Use exceptional isomorphism to replace $G = U(W)$ by another group $\tilde{G}$.
- Use theta correspondence to construct Π on $\tilde{G}$.
- Use Kudla-Millson theory to construct a differential form
- Use a seesaw diagram (in the sense of Kudla) to evaluate the period.

Relation with other work on the Hodge conjecture

- Bergeron-Millson-Moeglin : prove the Hodge conjecture for certain orthogonal and unitary Shimura varieties in certain degrees.
- Rough idea: show that all Hodge classes in these degrees are spanned by classes of special cycles (+ cycles coming from Lefschetz)
- Our work is orthogonal to this: we produce Hodge classes, where there is no obvious special cycle
- General phenomenon: there are many more Hodge classes on Shimura varieties than special cycles. Some of the most interesting Hodge classes (eg. the ones that may give functoriality) don't seem to come from special cycles.

Another (recent) application of this idea

- Recent work of Naomi Sweeting: ArXiv Dec 2022
- Start with pair (f, g) where f has weight 4, g has weight 2 and consider the Yoshida lift F to $\mathrm{GSp}(4)$.
- The Hodge structure attached to F is of type:

$$(3,0)+(2,1)+(1,2)+(0,3)$$

- The Galois rep is $V_f \oplus V_g(-1)$.
- Sweeting constructs Hodge class on $X \times Y$, where X is a modular curve and Y is a Siegel threefold, reflecting the fact that V_g occurs in both. In one special case, she also constructs a cycle.
- The analogous problem for f is open!

Next lecture

- Discuss the case of elliptic curves over imaginary quadratic fields (and more generally CM fields)
- Beilinson's conjectures on L-values at non-critical points will play a role
- Higher Chow groups instead of Chow groups
- The case of coherent cohomology