Stability in Trading Networks

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Networks of matching markets

- Many real-world matching markets are not two-sided. Rather they are networks of matching markets.
- Stability is an important equilibrium concept for analysis.
- In a supply chain, all firms are partially ordered e.g. farmer $\rightarrow$ supermarket $\rightarrow$ consumer.
  - A downstream firm never sells upstream.
- While this may be a good way to model an isolated industry, in general, production networks are interdependent and have *contract cycles*.
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Model

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- Set of firms (nodes) F and contracts (directed edges) X.
- Each contract $x \in X$ is bilateral involving a buyer and a seller: $F(x) \equiv \{b(x), s(x)\}$.

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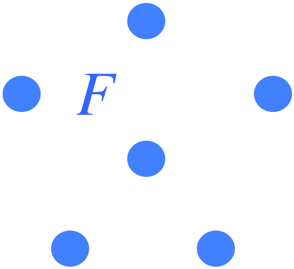
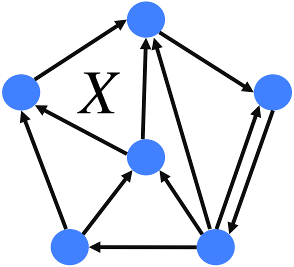
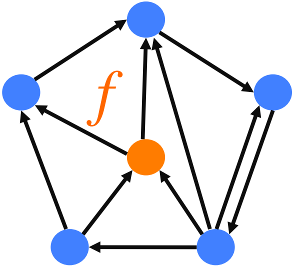
Firms



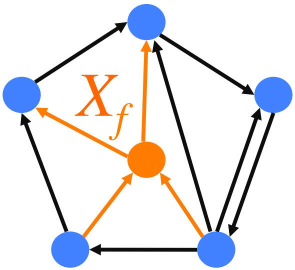
Contracts



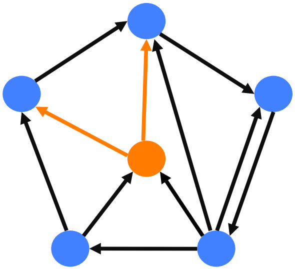
A firm



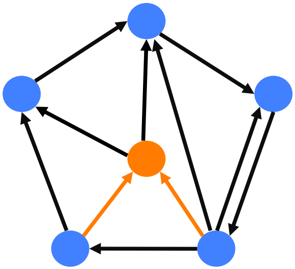
Firm f's contracts



Downstream contracts



Upstream contracts



Choice functions

- For any $Y \subseteq X$, the contracts that involve $f \in F$ are denoted Y_f. $F(Y)$ is the set of firms associated with contract set $Y \subseteq X$. Choice function is $C^f(Y_f) \subseteq Y_f$ for any $Y_f \subseteq X_f$.
- Choice functions C^f satisfy IRC if for any $Y \subseteq X$ and $C^f(Y) \subseteq Z \subseteq Y$ we have that $C^f(Z) = C^f(Y)$

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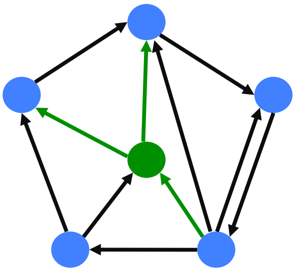
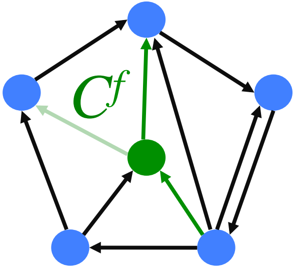
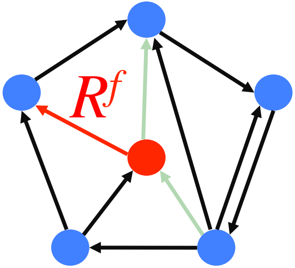
Offered contracts



Chosen contracts



Rejected contract



Reminder: substitutability

Definition

Preferences of $f \in F$ over contracts in X are *substitutable* if for all $Y' \subseteq Y \subseteq X$ we have $R^f(Y') \subseteq R^f(Y)$
in other words, $Y' \setminus C^f(Y') \subseteq Y \setminus C^f(Y)$

That is, if choosing from a bigger set of contracts, the rejected set expands.

Key assumption: full substitutability

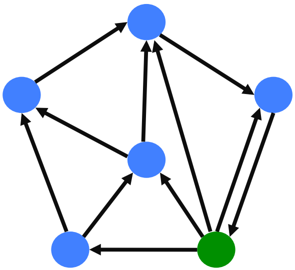
Definition

Preferences of $f \in F$ over contracts in X are *fully substitutable* if for all $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$ they are:

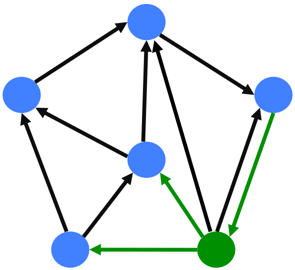
1. *Same-side substitutable* (SSS):
    1. $R^f_B(Y'|Z) \subseteq R^f_B(Y|Z)$
    2. $R^f_S(Z'|Y) \subseteq R^f_S(Z|Y)$
2. *Cross-side complementary* (CSC):
    1. $R^f_B(Y|Z) \subseteq R^f_B(Y|Z')$
    2. $R^f_S(Z|Y) \subseteq R^f_S(Z|Y')$

That is, if choosing from a bigger set of contracts on one side, by SSS the rejected set on the same side expands, and by CSC the chosen set on the opposite side expands.

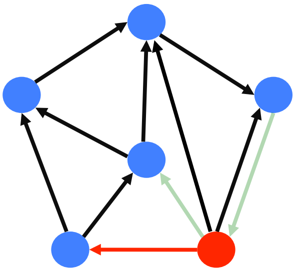
Same-side substitutability



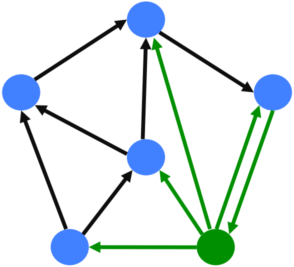
Same-side substitutability
Offered set



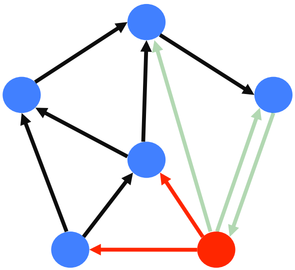
Same-side substitutability
Rejected set R^f



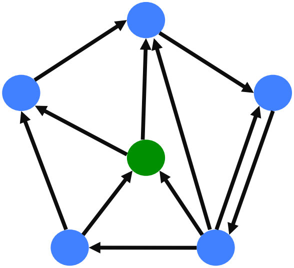
Same-side substitutability
More downstream contracts



Same-side substitutability
Continue rejecting from R^f



Cross-side complementarity



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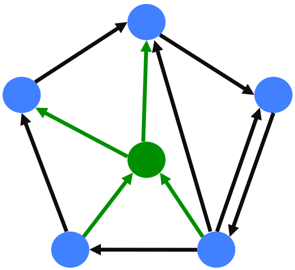

Cross-side complementarity
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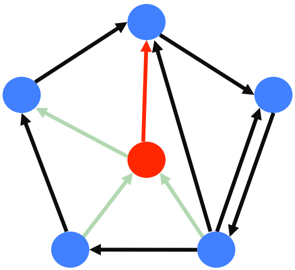
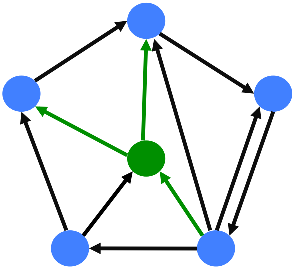
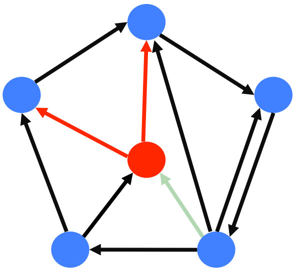


Cross-side complementarity
Fewer downstream contracts



Cross-side complementarity
Continue rejecting from R^f



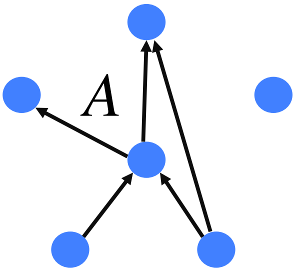
Stability concepts

What is a trail?

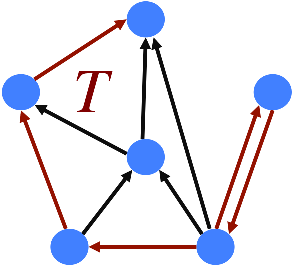
Definition

A non-empty set of contracts $T = \{x^1, \ldots, x^M\}$ is a *trail* if $b(x^m) = s(x^{m+1})$ for all $m = 1, \ldots, M-1$.

Allocation

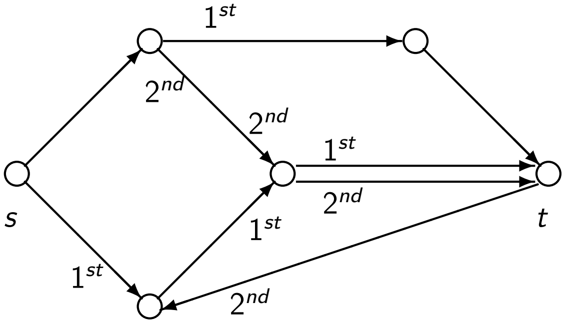


Trail



Special case

Here, the preferences are defined by a strict ordering over upstream and downstream contracts, and all firms except s and t satisfy Kirchhoff's law.



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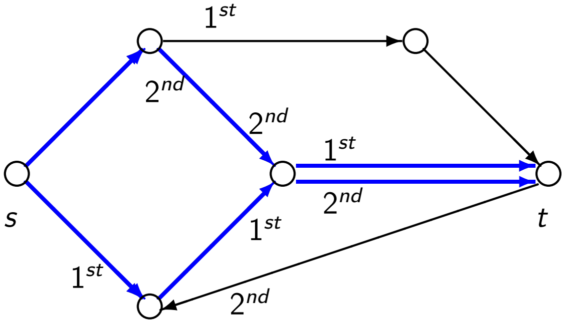

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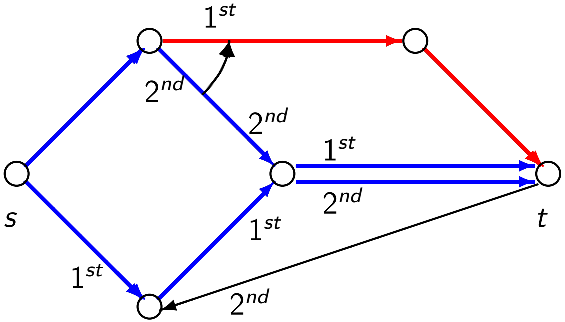

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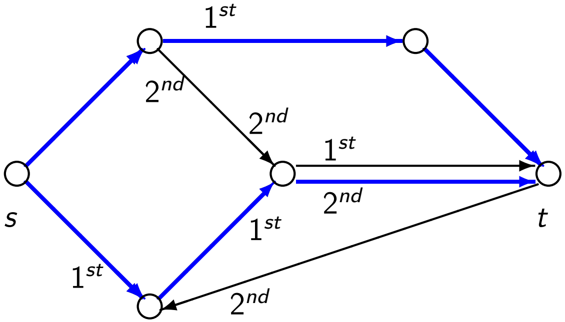

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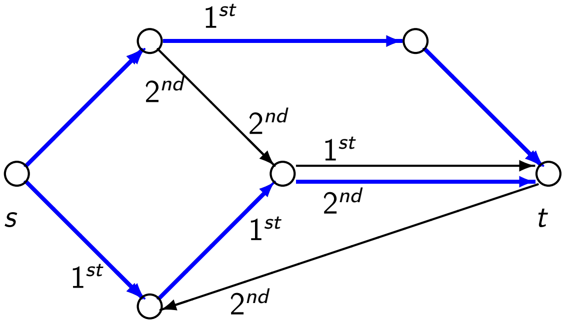
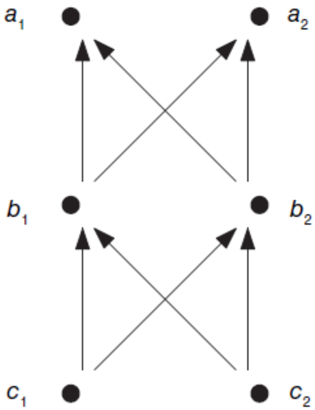


Supply chain



Chain-stability

Definition

An outcome $A \subseteq X$ is *chain-stable* if

1. *Individually rational*: for all $f \in F$, $C^f(A_f) = A_f$.
2. There is no trail $T \subseteq X$, such that $T \cap A = \emptyset$ and for all $f \in F(T)$, $T_f \subseteq C^f(A \cup T)$.

Acyclic supply chains

Theorem (Ostrovsky '08)

In any acyclic network X if preferences of F satisfy full substitutability and IRC then there exists a chain-stable outcome $A \subseteq X$.

Set stability

Definition

An outcome $A \subseteq X$ is *stable*[a] if it is:

1. *Individually rational*: for all $f \in F$, $C^f(A_f) = A_f$.
2. There is no non-empty **set** of contracts $Z \subseteq X$, such that $Z \cap A = \emptyset$ and for all $f \in F(Z)$, $Z_f \subseteq C^f(A \cup Z)$.

[a]Defined in Hatfield-Kominers '12

Theorem (Hatfield-Kominers '12)

Suppose that the set of contracts X is acyclic and that all firms' preferences are fully substitutable. Then an allocation A is stable if and only if it is chain stable.

Corollary

If the set of contracts X is acyclic and that all firms' preferences are fully substitutable, there exists a stable outcome $A \subseteq X$.

Theorem (Hatfield-Kominers-Nichifor-Ostrovsky-Westkamp '21)

In any trading network X if preferences of F satisfy full substitutability and IRC then the stable and chain-stable outcomes are the same.

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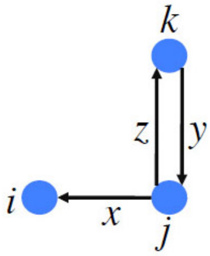
Set stability

- Preferences are fully substitutable:
  - $\prec_i$: $\{x\} \succ_i \emptyset$
  - $\prec_j$: $\{x, y\} \succ_j \{z, y\} \succ_j \emptyset$
  - $\prec_k$: $\{z, y\} \succ_k \emptyset$.
- No stable contract allocation exists.
- Theorem [FJTS]: Determining whether an outcome is stable is NP-complete.



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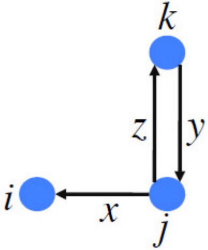

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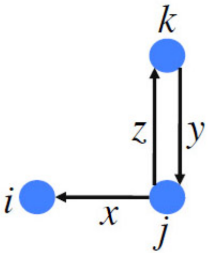


Trail stability

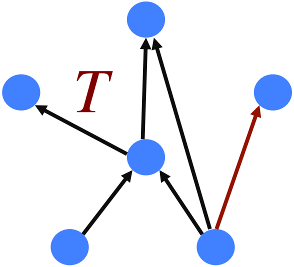
Definition

An outcome $A \subseteq X$ is *trail-stable* if it is:

1. *Individually rational*: for all $f \in F$, $C^f(A_f) = A_f$.
2. There is no locally blocking trail $T \subseteq X$, such that $T \cap A = \emptyset$ and
   1. For $f_1 = s(x^1)$, $x^1 \in C^{f_1}(A_{f_1} \cup x^1)$.
   2. For $f_m = b(x^{m-1}) = s(x^m)$, $\{x^{m-1}, x^m\} \subseteq C^{f_m}(A \cup x^{m-1}, x^m)$ for all $1 < m \leq M$.
   3. For $f_{M+1} = b(x^M)$, $x^M \in C^{f_{M+1}}(A_{f_{M+1}} \cup x^M)$.

A short-term stability notion. Trail stable outcomes do not require that the firm accept all its contracts along the trail. If production is sequential, the firm may know that it will only need to fulfil contracts further down the trail later. Divisions can make independent input-output decisions without overall firm consent.

Trail



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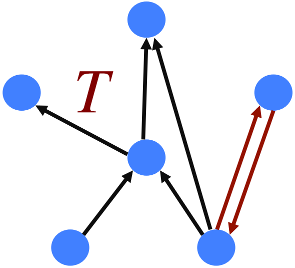

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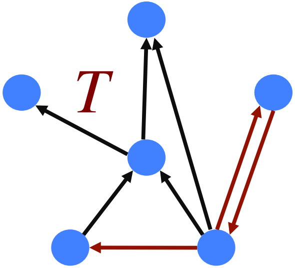

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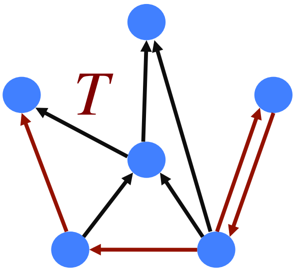

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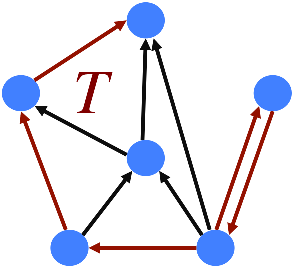

Trail stability – Existence

Theorem (FJTT)

In any contract network X if preferences of F satisfy full substitutability and IRC then there exists a trail-stable outcome $A \subseteq X$.

Side note: A partial order

Define a lattice L with the ground set $X \times X$ with an order $\sqsubseteq$ such that $(Y, Z) \sqsubseteq (Y', Z')$ if $Y \subseteq Y'$ and $Z \supseteq Z'$.

Existence – Sketch of Proof

Consider Y and Z, which are subsets of X,
Furthermore, define a mapping Φ as follows:
$\Phi_B(Y, Z) = X \setminus R_S(Z|Y)$
$\Phi_S(Y, Z) = X \setminus R_B(Y|Z)$
$\Phi(Y, Z) = (\Phi_B(Y, Z), \Phi_S(Y, Z))$

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Φ is isotone. We use Tarski's fixed point theorem, the fixed points of Φ from a lattice.
We claim that every fixed point (Y, Z) of Φ corresponds to an outcome $Y \cap Z = A$ that is trail-stable.

Alternative version with T

Define a pair of functions, $T_B(Z, Y)$ and $T_S(Z, Y)$, by
$T_B(Z, Y) := \{x \in X : x \in C_B(Y \cup \{x\}|Z)\}$ and
$T_S(Z, Y) := \{x \in X : x \in C_S(Z \cup \{x\}|Y)\}$
Then consider a pair of equations $Z = T_B(Z, Y)$ and $Y = T_S(Z, Y)$.

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T is isotone. We use Tarski's fixed point theorem.

Theorem (Adachi)

(a) If (Z, Y) is a fixed point of T , then $Z \cap Y$ is a trail-stable outcome and $Z \cap Y = C_B(Y|Z) = C_S(Z|Y)$.
(b) Suppose A is a trail-stable outcome. Then there is a pair of sets (Z, Y) such that (Z, Y) is a fixed point of T and $A = Z \cap Y = C_B(Y|Z) = C_S(Z|Y)$.

The (Z,Y) pair is not unique

Example

$$
\begin{aligned}
\succ_m &: \{w\} \succ_m \emptyset \\
\succ_i &: \{x\} \succ_i \emptyset \\
\succ_k &: \{z, y\} \succ_k \emptyset \\
\succ_j &: \{x, y, z, w\} \succ_j \{x, w\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \{z, y\} \succ_j \emptyset.
\end{aligned}
$$

and other outcomes are not acceptable. We use the T function.

(Z, Y) is a fixed point of T		A
Z	Y	$Y \cap Z$
$\{x, y, z, w\}$,	$\{x, y, z, w\}$	$\{x, y, z, w\}$
$\{x, w\}$	$\{x, y, z, w\}$	$\{x, w\}$
$\{x, y, z, w\}$	$\{x, w\}$	$\{x, w\}$
$\{x, w\}$	$\{x, w\}$	$\{x, w\}$

The (Z,Y) pair is not unique

Why is this a problem?
Going back to two-sided markets, where the choice functions are substitutable and IRC:

- If (Z, Y) is a fixed point of Φ then $A = Z \cap Y$ is stable
- If (Z, Y) is a fixed point of T then $A = Z \cap Y$ is stable
- If A is stable, a corresponding (Z, Y) fixed point of Φ exists, but it need not be unique
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So we had a nice structural result in two-sided markets which is lost in trading networks :(

Terminals

- An agent is a **terminal buyer** or **terminal seller** if he can only sign upstream or downstream contacts.
- Contracts signed by terminal agents in some trail-stable outcome are called **terminal-trail-stable**.
- An outcome is **buyer-optimal** (seller-optimal) if all terminal buyers (terminal sellers) unanimously prefer it to any other outcome in the set.

Buyer- and seller-optimal

Theorem (FJTT)

In any contract network X if preferences of F satisfy full substitutability and IRC then the set of trail-stable outcomes contains buyer-optimal and seller-optimal outcomes.

Terminal superiority

Consider $A, W \subseteq X$ which are individually rational for all terminal agents.

Definition

A is **seller-superior** to W (denoted by $A \succeq W$) if $C^f(A_f \cup W_f) = A_f$ for each terminal seller f and $C^g(A_g \cup W_g) = W_g$ for each terminal buyer g.
A is **buyer-superior** to W (denoted by $A \succeq' W$) if $C^f(A_f \cup W_f) = W_f$ for each terminal seller f and $C^g(A_g \cup W_g) = A_g$ for each terminal buyer g.

Of course, $A \succeq W$ if and only if $W \succeq' A$ holds. If either relation holds, we call this partial order **terminal superiority**.

Laws of Aggregate Demand and Supply

Definition

Preferences of $f \in F$ satisfy the Law of Aggregate Demand and the Law of Aggregate Supply, if for sets of contracts $Y' \subseteq Y \subseteq X_f^B$, and $Z \subseteq Z' \subseteq X_f^S$

$$|C_B^f(Y'|Z')| - |C_S^f(Z'|Y')| \leq |C_B^f(Y|Z)| - |C_S^f(Z|Y)|$$

Lattice for trail stability

Theorem (FJTT, Sublattice Theorem)

Suppose that choice functions satisfy full substitutability and LAD/LAS. Then the fixed points of $\Phi(Y, Z) = (X \setminus R_S(Z|Y), X \setminus R_B(Y|Z))$ *form a nonempty, complete sublattice of* $(2^X \times 2^X, \sqsubseteq)$.

Lattice for trail stability

Conjecture

In any contract network X if preferences of F satisfy full substitutability and LAD/LAS then the terminal-trail-stable contract sets form a lattice under terminal-superiority.

Theorem (FJTT)

In any contract network X if preferences of F satisfy full substitutability and LAD/LAS then the terminal-seller-trail-stable outcomes form a lattice for terminal sellers. (Similarly for terminal buyers)

Lattice for trail stability

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Initial and terminal segments

Define $T_f^{\leq m} = \{x^1, ..., x^m\} \cap T_f$ to be firm f's contracts out of first m contracts in the trail and $T_f^{\geq m} = \{x^m, ..., x^M\} \cap T_f$ are firm f's contracts out of the last $M - m + 1$ contracts.

Weak trail stability

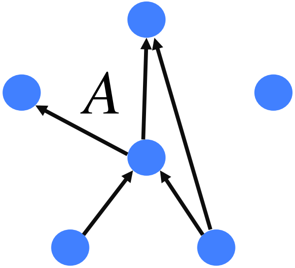
Definition

An outcome $A \subseteq X$ is *weak trail-stable* if it is:

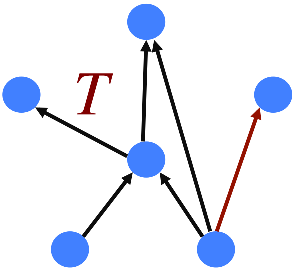
1. *Individually rational*: for all $f \in F$, $C^f(A_f) = A_f$.
2. *Trail-unblocked*: There is no trail $T \subseteq X$, such that $T \cap A = \emptyset$ and
   1. For $f_1 = s(x^1)$: $x^1 \in C^{f_1}(A_{f_1} \cup x^1)$, and
   2. For $f_m = b(x^{m-1}) = s(x^m)$: either $T^{\leq m}_{f_m} \subseteq C^{f_m}(A \cup T^{\leq m}_{f_m})$ or $T^{\geq m}_{f_m} \subseteq C^{f_m}(A \cup T^{\geq m}_{f_m})$ for all $1 < m \leq M$, and
   3. For $f_{M+1} = b(x^M)$: $x^M \in C^{f_{M+1}}(A_{f_{M+1}} \cup x^M)$.

Intuitively, a contract allocation is weak trail-stable if no agent wants to drop any of his contracts and there is no trail of contracts which the agents would choose *all contracts* **along** *the trail* instead of or with their allocation (weaker than HKNOW'15).

Allocation



Trail



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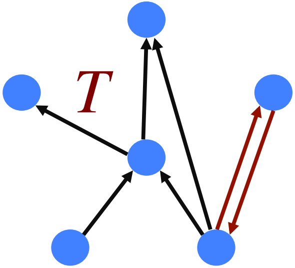
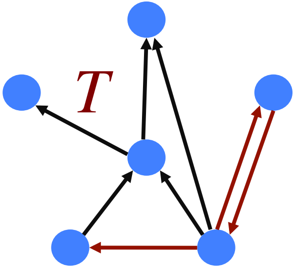
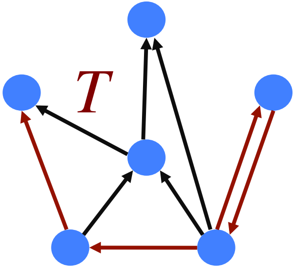


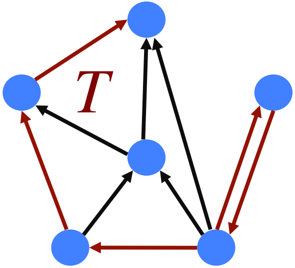
Trail



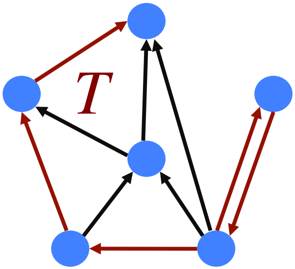
Trail



Trail

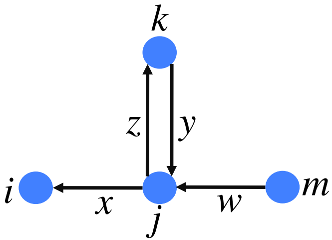


Blocking trail



Trail stability vs. Weak trail stability

- Preferences are fully substitutable:
  - $\prec_m$: $\{w\} \succ_m \emptyset$
  - $\prec_i$: $\{x\} \succ_i \emptyset$
  - $\prec_k$: $\{z, y\} \succ_k \emptyset$
  - $\prec_j$: $\{z, y\} \succ_j \{w, z\} \succ_j \{y, x\} \succ_j \emptyset$
- Weakly trail-stable: $\emptyset$ and $\{z, y\}$
- Trail-stable: $\{z, y\}$



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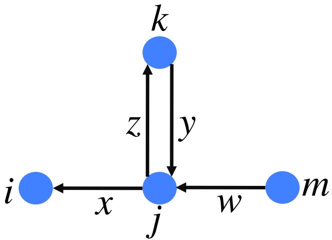

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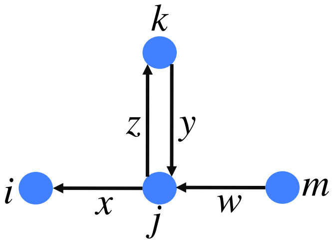

Trail stability vs. Weak trail stability

Lemma (FJTT)

In any contract network X where choice functions of F satisfy full substitutability and IRC, if A is a trail-stable outcome then A is also weakly trail-stable.

Corollary (FJTT)

In any contract network X if preferences of F satisfy full substitutability and IRC then a weakly trail-stable outcome $A \subseteq X$ exists.

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In any contract network X if preferences of F satisfy full substitutability and IRC then a weakly trail-stable outcome $A \subseteq X$ exists.

A new agent comes

We start from trail-stable outcome A. A new terminal seller enters the market. After a market readjustment process, they reach outcome A' which is trail-stable in the new scenario.

A new agent comes

Proposition

Consider a contract network X in which choice functions of F are fully substitutable and satisfy IRC. Suppose a new terminal seller f' whose choice function is fully substitutable and satisfies IRC enters the market.
Then each terminal seller $f \neq f'$ is prefers A to A' and each terminal buyer f is prefers A' to A.
The opposite holds when f' is terminal buyer.

Stability vs. trail stability

Are trail-stable outcomes always stable?

Stability vs. trail stability

Are trail-stable outcomes stable?

Of course not - they actually exist!

Consider a counterexample again...

Stability vs. trail stability

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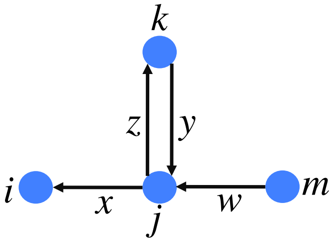
- No stable contract allocation exists.
- However, there is a trail-stable allocation $A = \{w\}$.



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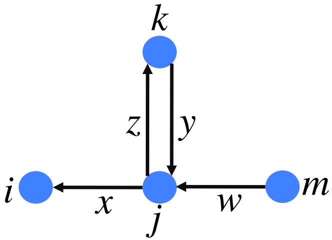

Path-or-cycle stability

Definition

An outcome $A \subseteq X$ is *path-or-cycle-stable* if

1. A is individually rational.
2. There is no path or cycle B such that $B \cap A = \emptyset$ and $B_f \subseteq C^f(B_f \cup A_f)$ for each $f \in F(B)$. Such paths or cycles are called *blocking paths* and *blocking cycles*.

Stability in flow networks

An interesting property of flow-based choice functions is that given an outcome $A \subseteq X$, any cycle C disjoint from A is a blocking cycle, as any firm which is offered a pair of additional upstream and downstream contracts will accept them.

Theorem (FJST)

Suppose that in a flow network choice functions are flow-based. Then it is NP*-complete to decide if the flow network admits a path-or-cycle-stable outcome.*

Stability in flow networks

We next prove that in flow networks path-or-cycle-stable outcomes coincide with stable outcomes.

Theorem (FJST)

In a flow network an outcome is path-or-cycle-stable if and only if it is stable.

Corollary (FJST)

It is NP*-complete to decide if a flow network admits a stable outcome.*

Back to general trading networks

Corollary

Suppose that in a trading network choice functions satisfy full substitutability and IRC. Then it is NP*-hard to decide if the trading network admits a stable outcome.*

A decision problem: An instance of *Instability* is a trading network and an outcome A. The answer for an instance of Instability is YES if the particular outcome A is not stable.

Theorem

The Instability *problem is NP-complete. Moreover, if choice functions are represented by oracles, then finding the right answer for an instance of* Instability *might require an exponential number of oracle calls.*

Conclusion

How many stability definitions did we see today?

- chain-stable
- stable
- trail-stable
- weak trail-stable
- path-or-cycle stable
- In an acyclic supply chain, these are all the same.

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- A stable outcome is always trail-stable, but not the other way around.
- Trail-stable: always exists, easy to find.
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Thank you for your attention!

Köszönöm a figyelmet!