Recent Developments in the Analysis of Markets for Indivisible Goods

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Oxford

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General model of an exchange economy

Ingredients:

- finite set N of goods,
- finite set J of agents,
- X is the set of feasible consumption bundles for each agent,
- agent j's utility function is $U^j(\mathbf{x})$ for $\mathbf{x} \in X$,
- agent j's endowment $\mathbf{w} \in X$ is feasible,
- prices are $\mathbf{p} \in \mathbb{R}^N$,
- $D^j_{\mathrm{M}}(\mathbf{p}, \mathbf{w})$ is the (Marshallian/uncompensated) demand correspondence, i.e., set of feasible consumption bundles that maximize agent j's utility .

definition

given endowments $(\mathbf{w}^j)_{j \in J}$, pair $((\mathbf{x}^j)_{j \in J}, \mathbf{p})$ is a *competitive equilibrium* if $\mathbf{x}^j \in D^j_{\mathrm{M}}(\mathbf{p}, \mathbf{w}^j)$ for each $j \in J$ and $\sum_{j \in J} \mathbf{x}^j = \sum_{j \in J} \mathbf{w}^j$.

Markets for divisible goods and general equilibrium

Suppose that for all $j \in J$:

- goods are divisible $X \subseteq \mathbb{R}^N$.
- $U^j(\mathbf{x})$ is continuous, monotone/locally non-satiated, and (quasi-)concave;
- endowments are positive, i.e., $\mathbf{w}^j \gg 0$ for all $j \in J$.

Markets for divisible goods and general equilibrium

1. Equilibrium exists (Arrow and Debreu (1954); McKenzie (1954))
2. Generically, the number of equilibrium price vectors is finite (Debreu, 1970) and odd (Dierker, 1972).
3. Anything goes: aggregate demand places few restrictions on individual demands (Sonnenschein (1973), Mantel (1974), Debreu (1974)) and equilibrium entails very few restrictions on the set of equilibrium prices (Mas-Colell, 1977).
4. The core (set of unblocked allocations) is larger than the set of competitive equilibrium allocations.
5. Tâtonnement (myopic price adjustment) works only under stronger conditions on preferences (Arrow and Hurwicz, 1958).
6. Computing equilibrium prices precisely is hard (Scarf, 1973; Papadimitriou, 1994, Chen et al., 2008).

Markets for indivisible goods

MWG (p. 598):

> *the most substantial [assumption for the existence of equilibrium] concerns convexity"*

Many markets are thin and involve trade of highly heterogeneous goods. Indivisibilities can play a important role in. . .

- **exchange**: housing markets, markets for used cars. . .
- **auctions**: spectrum auctions, ad slots. . .
- **labour markets**: specialized jobs. . .
- **production**: highly specific inputs, machines. . .

Model for markets with indivisible goods

For the rest of the lecture assume the following:

- all goods except one good x_0 called "money" (the numeraire) are **in**divisible;
- set I of indivisible goods;
- $X_I \subseteq \mathbb{Z}^I$ of feasible bundles of indivisible goods;
- $\mathbf{x} = (x_0, \mathbf{x}_I) \in \mathbb{R} \times X_I = X$ are feasible consumption bundles.

Transferable Utility Economies

Transferable utility

- We will assume that $U^j(\mathbf{x}) = V^j(\mathbf{x}_I) + x_0$ for some *valuation* $V^j : X_I \to \mathbb{R}$ and money $x_0 \in \mathbb{R}$.
- Efficient outcomes are found by maximizing the sum of valuations.
- Endowments do not affect demand, so we can write demand as $D^j(\mathbf{p})$.

Exchange economy $\leftrightarrow$ two-sided market

- Redefine J to be "buyers" who have the same utility function as before but own nothing.
- Redefine I to be "sellers" who have a zero utility function and own goods I.

proposition (Bikhchandani and Mamer, 1997; Ma, 1998)

competitive equilibrium exists in the exchange economy if and only if it exists in the modified two-sided market.

Assignment market

- Suppose that $X_I = \{\mathbf{0}\} \cup \{\mathbf{e}^i | i \in I\}$, i.e., there are multiple heterogeneous goods, but any agent can consume at most one good.

Assignment market

- Denote by $v_{ij} \geq 0$ the surplus created good i (owned by seller i) is bought by buyer j.
- Denote by α_{ij} the (fractional) assignment of good i to agent j. What's the efficient assignment?

$$\max_{i,j} \quad \sum_i \sum_j v_{ij}\alpha_{ij}$$

$$\text{s.t.} \quad \sum_i \alpha_{ij} = 1 \quad \text{for all } j \in J,$$

$$\sum_j \alpha_{ij} = 1 \quad \text{for all } i \in I,$$

$$\alpha_{ij} \geq 0 \quad \text{for all } i \in I \text{ and } j \in J.$$

- This problem must have an integral solution. Why?

That's why

T. C. KOOPMANS AND M. BECKMANN

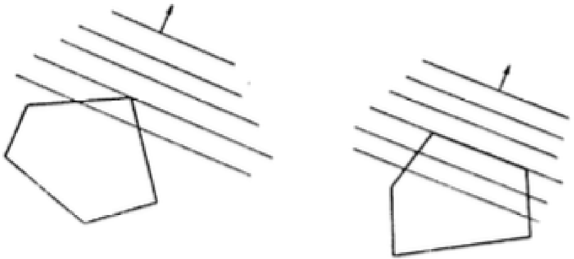


FIGURE 1.—Maximum of a Linear Function on a Polyhedron.

Assignment market

theorem (Koopmans and Beckmann, 1957)

there exists a competitive equilibrium in the assignment market.

primal gives us the allocation, dual gives us the prices.

theorem (Shapley and Shubik, 1971)

competitive equilibrium outcomes coincide with the core.

the constraints in the dual gives feasibility and objective gives non-improvability.

theorem (Shapley and Shubik, 1971)

there exist minimum-price and maximum-price competitive equilibria.

more generally there is a lattice of equilibrium prices.

Competitive equilibrium with multiple goods

Denote by $\mathbf{y}_I$ the vector of total endowment of indivisible goods.

$$\text{LPRIP} = \max_{(\alpha^j)_{j\in J}} \quad \sum_{j\in J} \sum_{\mathbf{x}_I^j \in X_I} \alpha^j_{\mathbf{x}_I^j} V^j(\mathbf{x}_I^j)$$

$$\text{s.t.} \quad \sum_{\mathbf{x}_I^j \in X_I} \alpha^j_{\mathbf{x}_I^j} = 1 \quad \text{for all } j \in J,$$

$$\sum_{j\in J} \sum_{\mathbf{x}_I^j \in X_I} \alpha^j_{\mathbf{x}_I^j} \mathbf{x}_I^j = \mathbf{y}_I,$$

$$\alpha^j \in \mathbb{R}^{X_I}_{\geq 0} \alpha^j \in \{0,1\}^{X_I} \quad \text{for all } j \in J.$$

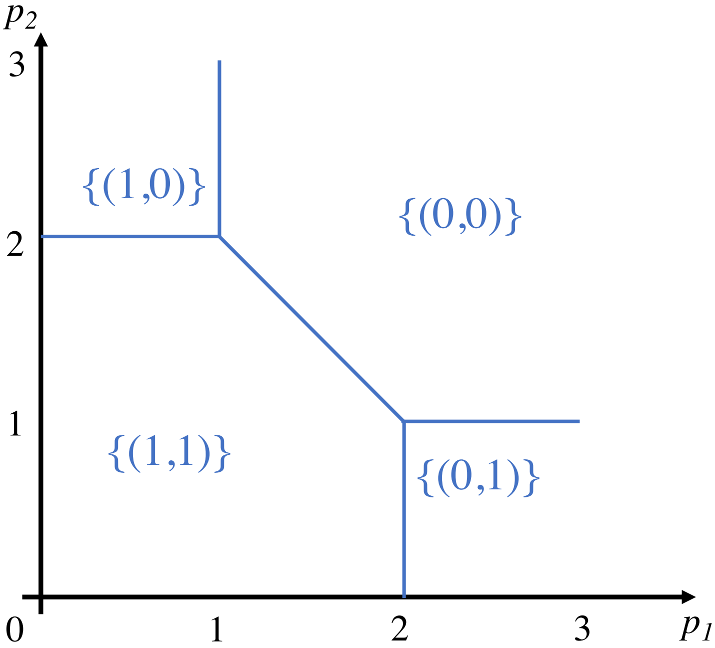
theorem (Bikhchandani and Mamer, 1997)

competitive equilibrium exists if and only if the values of the optimal solutions to the IP and LPR coincide.

Multi-good demand

- From now on and until the rest of the talk assume that $X_I = \{0,1\}^I$: agents might want multiple goods but **at most one unit of any good**.
- Suppose there two goods. Seller owns both goods and values them at nothing. There are two buyers j and k with the following valuations.

$\mathbf{x}_I$	$(1,0)$	$(0,1)$	$(1,1)$
$V^j(\mathbf{x}_I)$	1	1	3
$V^k(\mathbf{x}_I)$	2	2	3



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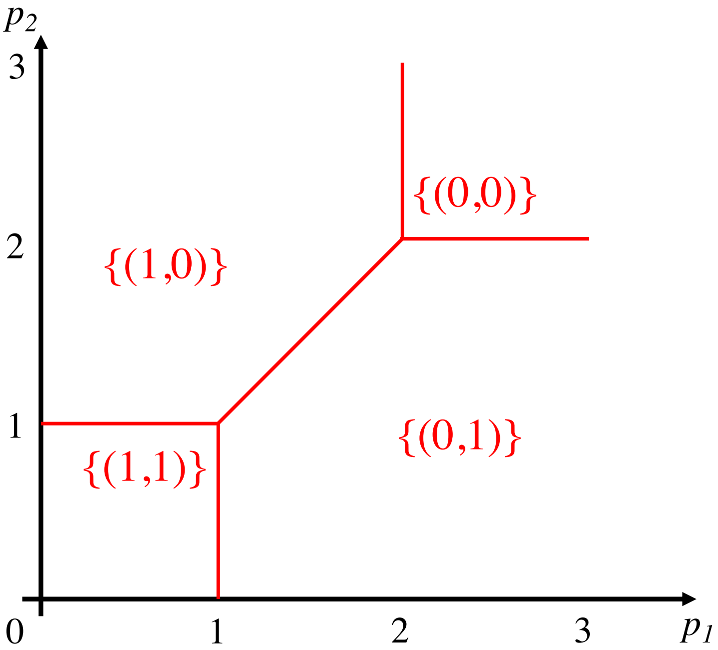

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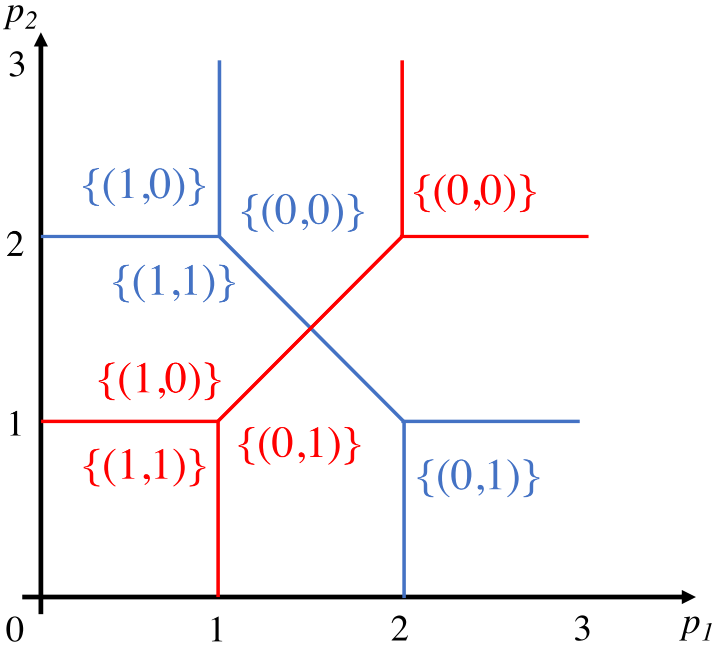

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Substitutes

- The following assumption on preferences is important.

definition (Kelso and Crawford, 1982; Ausubel and Milgrom, 2002)

a valuation V^j is a *substitutes valuation* if for all price vectors $\mathbf{p}_I$ and $\lambda > 0$, whenever $D^j(\mathbf{p}) = \{\mathbf{x}_I\}$ and $D^j(\mathbf{p} + \lambda \mathbf{e}^i) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.

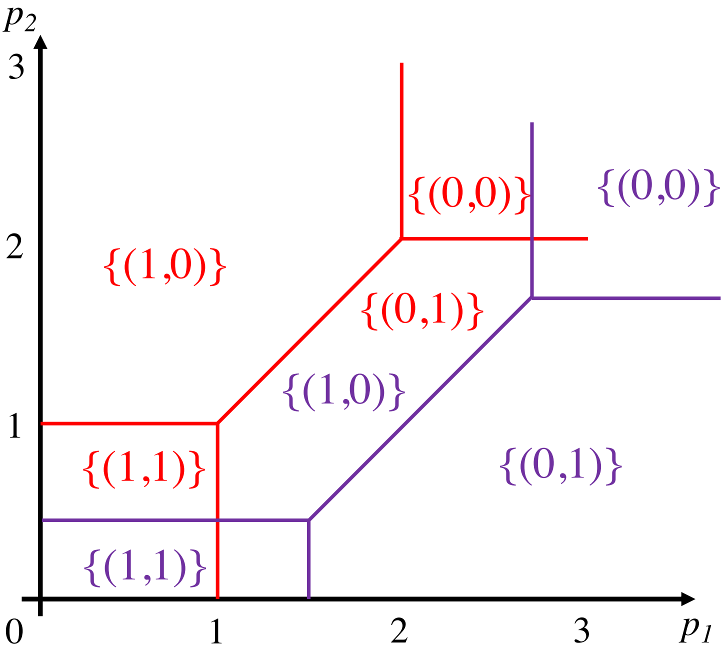
theorem (Kelso and Crawford, 1982)

if all agents have substitutes valuations, then competitive equilibria exist.

Existence under substitutes

- Discretize prices, so demand is always single-valued on the grid.[1]
- Start prices very low. If more than one firm demands the good, increase its price.
- By substitutability, as prices of some goods rise, demand for other goods weakly increases.
- Eventually, the market for each good clears.
- Take limits, obtain equilibrium in the economy with continuous prices.

[1] Not necessary but slightly fiddlier, see Gul and Stachetti (1999).



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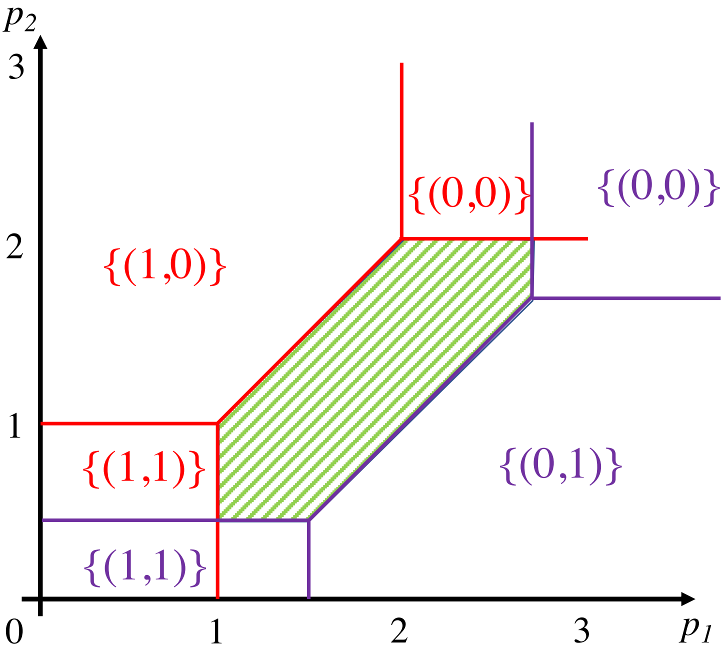
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Facts about substitutes (Gul and Stachetti, 1999)

- Tight connection between monotonic auctions and Deferred Acceptance Algorithm of Gale and Shapley (1962).
- Equilibrium prices form a (complete) lattice.
- Ascending/descending auction finds the lowest/highest equilibrium prices.
- In a large enough replica economy, lowest equilibrium prices "coincide" with Vickrey-Clarke-Groves payments.
- Substitutability forms a maximal domain of preferences for existence of equilibrium.

theorem (∼Gul and Stachetti, 1999)

If $|J| \geq 2$, agent j demands at most one unit of each good, and V^j is not a substitutes valuation, then there exist substitutes valuations V^j for agents $k \neq j$ for which no competitive equilibrium exists.

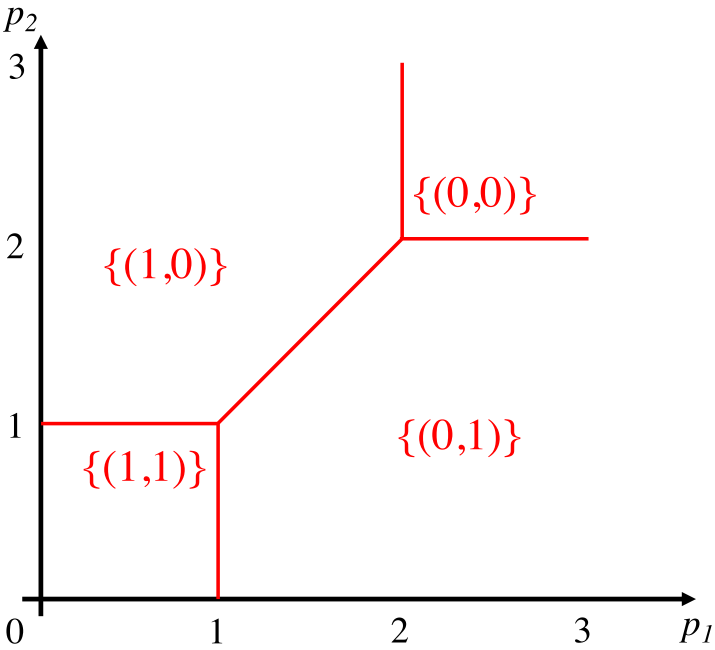
Demand types

- Danilov, Koshevoy and Murota (2001) and Baldwin and Klemperer (2019) used beautiful methods from combinatorial and tropical geometry to describe preferences and study equilibrium.
- Building on Baldwin and Klemperer's "demand types", it turns out to be especially easy to describe comparative statics of demand very generally when agents want/there is at most one unit of each good!
- Let $\mathcal{D} \subseteq \mathbb{Z}^I$ be a set of integer vectors.

definition (Baldwin, Jagadeesan, Klemperer, T., 2022wp)

valuation V^j is of demand type $\mathcal{D}$ if for all goods i, price vectors $\mathbf{p}_I$ and $\mathbf{p}'_I$ such that $p'_i > p_i$, with $D(\mathbf{p}) = \{\mathbf{x}_I\}$ and $D(\mathbf{p}') = \{\mathbf{x}'_I\}$, the price effect $\mathbf{x}'_I - \mathbf{x}_I$ is either 0 or an element of $\mathcal{D}$.

- if two goods are substitutes: $\mathcal{D} = \{(1,0),(0,1),(1,-1)\}$.



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Demand types continued

- a demand type $\mathcal{D}$ is unimodular if any linearly independent set of vectors in $\mathcal{D}$ is an integer basis for the subspace they span.

theorem

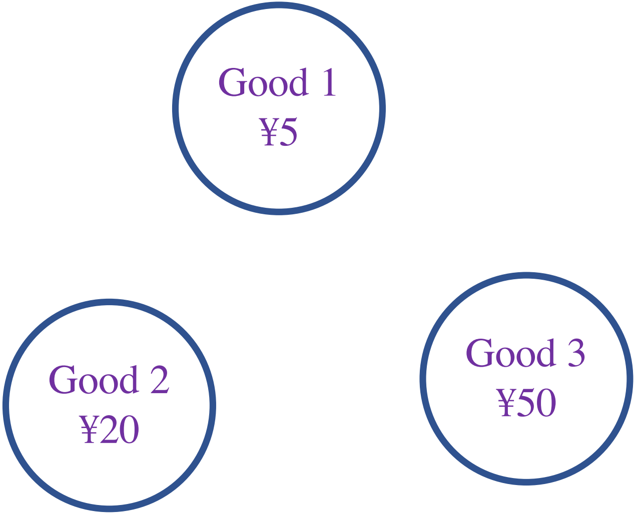
if $\mathcal{D}$ is unimodular and all agents' valuations are of demand type $\mathcal{D}$, then competitive equilibrium exists.

- example of a unimodular demand type that is not a unimodular basis change of substitutes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Beyond demand types: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".


Good 1
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Good 2
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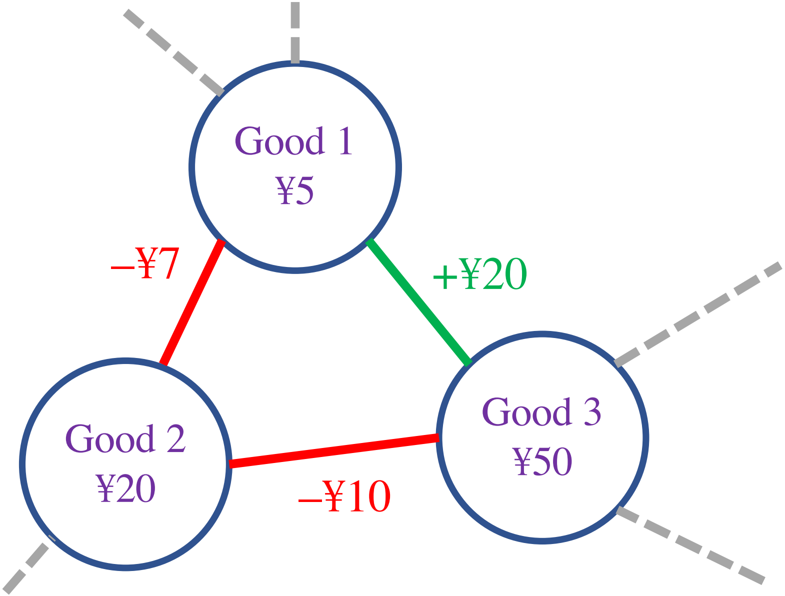
Good 3
¥50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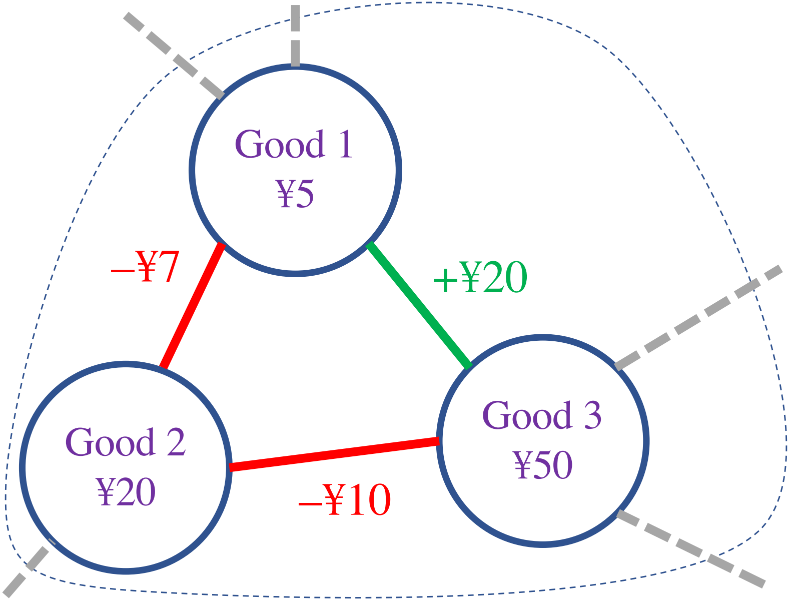

Good 1
¥5
–¥7
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Good 2
¥20
Good 3
¥50
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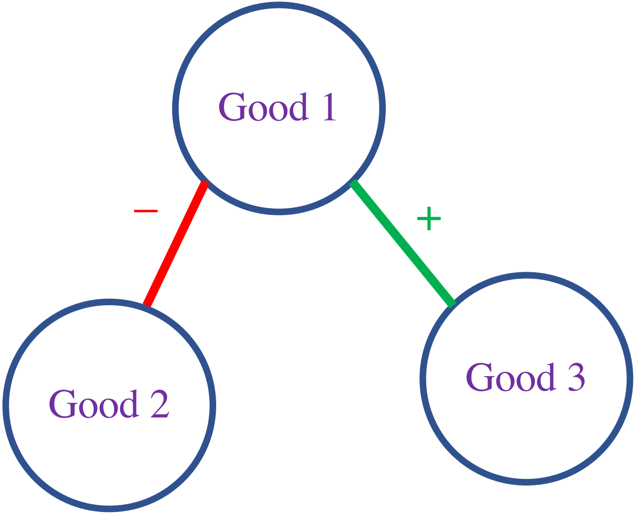
Beyond demand types: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".
- Assume, moreover, than the graph is a tree and is sign-consistent.


Good 1
−
+
Good 2
Good 3

Beyond demand types: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".
- Assume, moreover, than the graph is a tree and is sign-consistent.

theorem (Candogan, Ozdaglar, and Parrilo, 2015)

if all agents have sign-consistent tree valuations, then competitive equilibria exist.

Income effects

Model: exchange economy

for each agent j:

- still at most one unit of each good, $\mathbf{x}_I \in \{0,1\}^I$, but now $x_0 \in (\underline{m}, \infty)$
- endowment of goods + money is $\mathbf{w} = (w_0, \mathbf{w}_I)$ so $x_0 = w_0 - \mathbf{p}_I \cdot \mathbf{x}_I$
- utility function $U^j : (\underline{m}, \infty) \times \{0,1\}^I \to (\underline{u}^i, \overline{u}^i)$ is continuous and increasing in x_0 for each $\mathbf{x}_I$, and satisfies

$$\lim_{x_0 \to \underline{m}} U^j(\mathbf{x}_I, x_0) = \underline{u}^j \quad \text{and} \quad \lim_{x_0 \to \infty} U^j(x_0, \mathbf{x}_I) = \overline{u}^j$$

- where $\underline{m} \geq -\infty$ rules out "hard" budget constraints...
- ...but in their presence use "~~competitive~~ quasi-equilibrium" (Debreu, 1962)

Examples of preferences

example (quasilinear utility "without a budget constraint")

$$U^j(\mathbf{x}) = V^j(\mathbf{x}_I) + x_0$$

where $V^j(\mathbf{x}_I)$ is a valuation.

example (additively separable utility with "soft" budget constraint)

$$U^j(\mathbf{x}) = f(\mathbf{x}_I) + g(x_0)$$

for some regularity conditions on g, satisfied by, e.g., "quasilog" utility

$$U^j(\mathbf{x}) = \log(x_0) - \log(-V^j_{\mathrm{Q}}(\mathbf{x}_I))$$

where $V^j_{\mathrm{Q}} : \{0,1\}^I \to (-\infty, 0)$ is a "quasivaluation".

Marshallian and Hicksian demand

- Marshallian demand $D^j_{\mathrm{M}}(\mathbf{p}_I; \mathbf{w})$ is a solution to:

  max $U^j(\mathbf{x})$ given endowment $\mathbf{w}$ and prices $\mathbf{p}_I \in \mathbb{R}^I$.

- Hicksian demand $D^j_{\mathrm{H}}(\mathbf{p}_I; u)$ is a solution to:

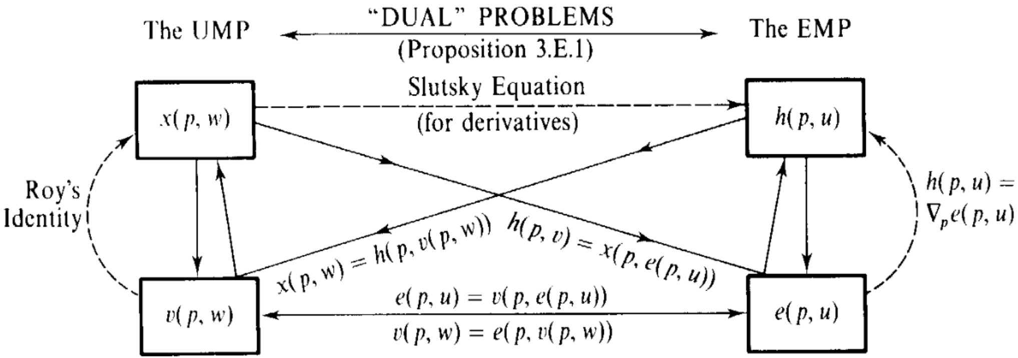
  min $\mathbf{p} \cdot \mathbf{x}$ given utility level u and prices $\mathbf{p}_I \in \mathbb{R}^I$.

- here, as in classical demand theory, there is a Marshallian-Hicksian duality

- for quasilinear preferences: $D^j_{\mathrm{M}}(\mathbf{p}_I; \mathbf{w}) = D^j_{\mathrm{H}}(\mathbf{p}_I; u)$, so we write

$$D^j(\mathbf{p}_I) = \arg\max_{\mathbf{x}_I \in X_I} \left\{ V^j(\mathbf{x}_I) - \mathbf{p}_I \cdot \mathbf{x}_I \right\}$$

Textbook Summary



Quasilinear interpretation of Hicksian demand

- write $S^j(\mathbf{x}_I; u) = U^j(\cdot, \mathbf{x}_I)^{-1}(u)$ for the money to get utility u given $\mathbf{x}_I$
  - this is the "compensation function" of Demange and Gale (1985)

definition

for a utility level u, the *Hicksian valuation* of agent j is $V^j_{\mathrm{H}}(\mathbf{x}_I; u) = -S^j(\mathbf{x}_I; u)$

lemma

for all price vectors $\mathbf{p}_I$ and utility levels u, we have

$$D^j_{\mathrm{H}}(\mathbf{p}_I; u) = \arg\max_{\mathbf{x}_I \in X_I} \{V^j_{\mathrm{H}}(\mathbf{x}_I; u) - \mathbf{p}_I \cdot \mathbf{x}_I\}$$

- the Hicksian valuations at fixed u captures substitution effects, while variation in the Hicksian valuations across u captures income effects

The Hicksian economies

definition

- for a utility level u, the *Hicksian valuation* of agent j is $V^j_{\mathrm{H}}(\cdot; u)$
- for a profile $(u^j)_{j \in J}$ of utility levels, the *Hicksian economy* is the TU economy in which agent j's valuation is her Hicksian valuation for u^j

- lemma $\implies$ demand in Hicksian econ. is Hicksian demand in original

- by construction, no income effects in the Hicksian economies
- price effects in each Hicksian economy are substitution effects

Books by J. R. Hicks in Don Patinkin's library at Hebrew U.

Example: housing market with endowments

- assignment game: allocating objects to unit-demand agents with quasilinear utility (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971)
- housing market: exchanging houses among unit-demand agents with endowments + income effects (Quinzii, 1984; Gale, 1984; Svensson, 1984)

Example: housing market with endowments

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assignment game ←- - - quasilinear interpretation of Hicksian demand - - -→ housing market

Equilibrium existence duality

- fixing supply of indivisibles, assume agents' endowments are feasible consumption bundles

Equilibrium existence duality

- fixing supply of indivisibles, assume agents' endowments are feasible consumption bundles

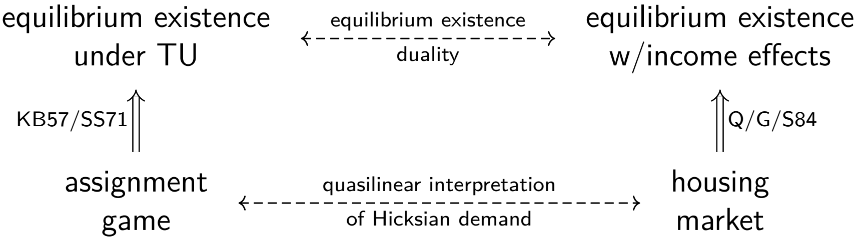
theorem (Equilibrium Existence Duality)

competitive equilibria exist for all utility profiles in *all* the Hicksian economies	$\Longleftrightarrow$	competitive equilibria exist for all endowment profiles in the original economy

- interpretation: *substitution effects* determine whether equilibrium exists
  - since each Hicksian economy (LHS) only contains substitution effects
- $\Longleftarrow$: basically Maskin and Roberts's (1980) proof of 2nd Welfare Theorem
  - competitive equilibria in the Hicksian economy $\sim$ quasiequilibria w/transfers
- $\Longrightarrow$: fixed-point argument using utility levels ($\sim$ Luenberger, 1994)

Example: housing market with endowments

- housing market: exchanging houses among unit-demand agents with endowments + income effects (Quinzii, 1984; Gale, 1984; Svensson, 1984)
- assignment game: assigning objects to unit-demand agents with quasilinear preferences (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971)



Substitutability conditions

definition (∼Kelso and Crawford, 1982)

U^j is *gross substitutable at an endowment* $\mathbf{w}$ if for that endowment,
raising the price of a good never lowers Marshallian demand for any other good

- under gross substitutability, competitive equilibria exist and can be found by a simultaneous ascending auction (+ lattice structure, incentives, ...)
  - Kelso and Crawford (1982), Gul and Stacchetti (1999), Milgrom (2000), Fleiner, Jagadeesan, Jankó, and T. (2019), Schlegel (2022), ...

definition

U^j is *net substitutable* if for all utility levels,
raising the price of a good never lowers Hicksian demand for any other good

- for quasilinear utility functions, gross substitutability $\equiv$ net substitutability

Net substitutability is weaker than gross substitutability

- housing example has net substitutability but not gross substitutability
  - suppose Martine owns ♣ and is considering selling her house and buying either a luxurious house ♢ or a simple house ♠
  - if she only wants to buy ♢ if she will have enough money left over, then increases in the price of ♣ can make Martine stop demanding ♠

proposition

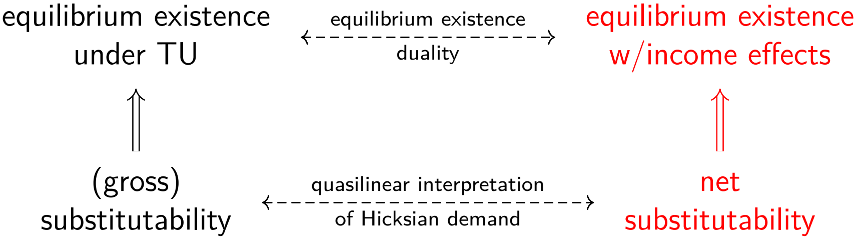
if there is a goods endowment $\mathbf{w}_I$ such that, for all money endowments w_0, U^j is gross substitutable at endowment $\mathbf{w}$, then U^j is net substitutable

- intuitively: gross substitutability constrains income and substitution effects, while net substitutability only constrains substitution effects
- result relies on indivisibility, not based on Slutsky equation

Net substitutability and the existence of competitive equilibrium

(gross) substitutability ←- - - quasilinear interpretation of Hicksian demand - - -→ net substitutability

Net substitutability and the existence of competitive equilibrium



corollary

under net substitutability, competitive equilibria exist for all endowment profiles

Net substitutability and the existence of competitive equilibrium

equilibrium existence under TU ←- - - equilibrium existence duality - - -→ equilibrium existence w/income effects

⇑ ⇑

(gross) substitutability ←- - - quasilinear interpretation of Hicksian demand - - -→ net substitutability

corollary

under net substitutability, competitive equilibria exist for all endowment profiles

- unlike under gross substitutability, simultaneous ascending auctions *may not* find equilibrium under net substitutability (+ straightforward bidding)
  - raising prices of overdemanded goods can cause underdemand of other goods

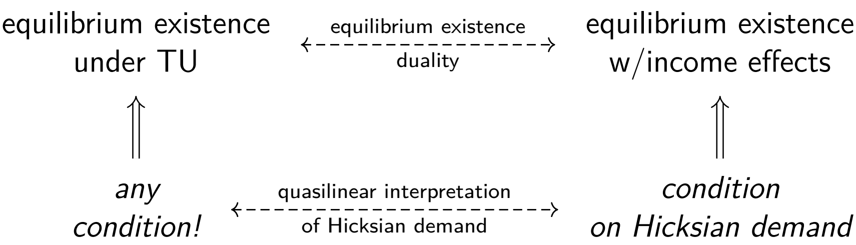
Net substitutability as a maximal domain

- we can use the $\Longleftarrow$ direction of the Equilibrium Existence Duality to prove a maximal domain result for an economy with income effects

corollary

suppose $|J| \geq 2$ and there is one unit of each good. if one agent does not have a net substitutes utility function, then there exist substitutes valuations for other agents such that there is no competitive equilibrium at some endowment profile.

The power of the Equilibrium Existence Duality



- *Any condition* on equilibrium existence in TU economies has a corresponding condition on Hicksian demands that guarantees equilibrium existence in an economy with income effects
- Examples:
  - integer-programming formulation (Bikhchandani and Mamer, 1997)
  - sign-consistent tree valuations (Candogan, Ozdaglar, and Parrilo, 2015)
  - unimodular demand types (BK, 2019; BK-Edhan-JT, 2022)
  - complements (Rostek and Yoder, 2020)

Markets for indivisible goods: Income effects vs. TU

1. Equilibrium requires strong assumptions on preferences.
2. Typically, a continuum of equilibrium prices.
3. Equilibrium prices lack structure.
4. The core coincides with competitive equilibrium allocations.
5. Tâtonnement does not work.
6. Computing equilibrium prices is hard.

Further directions

- equilibrium existence conditions at given endowment: Δ-substitutes (Nguyen and Vohra, 2022)
- hard budget constraints in matching markets (Jagadeesan and T., 2022)
- sealed-bid, near-feasible auction design for substitutes (à la Milgrom (2009) or Klemperer (2010)) with budget constraints (Nguyen and T., in progress)
- duality for markets vs. pseudomarkets (Nguyen and T., draft to share)
- complexity of finding equilibria (Lock, Qui, and T., in progress)
- frictions? networks? incentives? large markets?...

Thank you!

Graduate course MATH272 ``Market Design''
is at 9:30-11 every Tue&Thu in Evans 732

Credit enrolment deadline is today!

As of 1.31pm there were still 2 open seats!