Free boundary clusters with two phases

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I. The one-phase problem

II. The two-phase problem

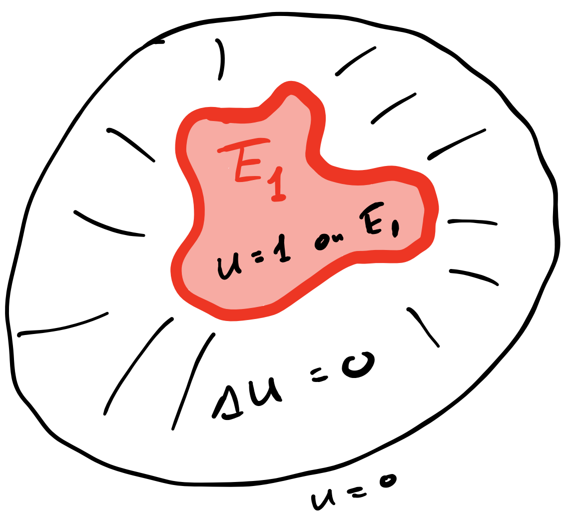
III. Free boundary clusters

(Bernoulli, Alt-Caffarelli)

I. The one-phase problem

E_1 smooth, bounded

$\mathbb{R}^d$



Bernoulli:

Find $u : \mathbb{R}^d \to \mathbb{R}$, $u \geq 0$

$$\begin{cases} u = 1 & \text{on } E_1 \\ \Delta u = 0 & \text{on } \{u>0\} \setminus E_1 \\ |\nabla u| = 1 & \text{on } \partial\{u>0\} \end{cases}$$

Find both u and $\Omega_u = \{u>0\}$

Alt-Caffarelli '1981

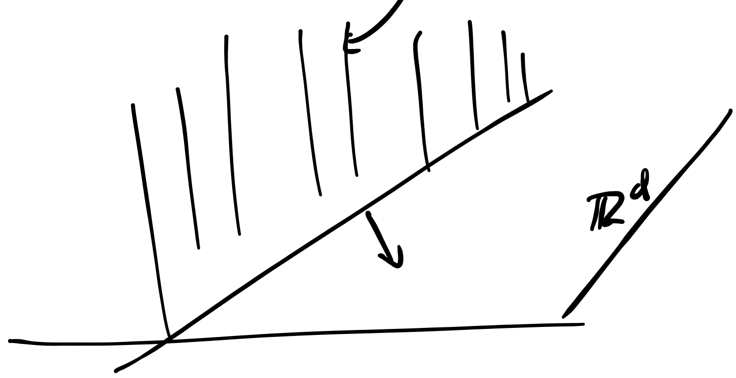
Minimize $$F(u;\mathbb{R}^d) = \boxed{\int_{\mathbb{R}^d} |\nabla u|^2 + \underbrace{|\{u>0\}|}}$$

$u \in H^1(\mathbb{R}^d)$, $u \equiv 1$ on E_1.

Flat free boundaries are $C^{1,\alpha}$.

↳ close in L^∞-norm to the half-plane sol.

$h(x) = (x \cdot \underset{\partial B_1}{\underset{\cap}{\nu}})_+$



1. The one-phase Bernoulli problem - biblio

Epsilon-regularity.

1981. H. W. Alt, and L. A. Caffarelli. *J. Reine Angew. Math.*. → flat $\Rightarrow C^{1,\alpha}$

2011. D. De Silva. *Interfaces and Free Boundaries.*

1977. D. Kinderlehrer, L. Nirenberg. *Ann. Scuola Norm. Sup. Pisa.*

$$\begin{cases} C^{1,\alpha} \Rightarrow C^{\infty} \\ C^{1,\alpha} \Rightarrow C^{\omega} \text{ in 2D} \end{cases}$$

Monotonicity formulas, homogeneity of the blow-ups, dimension reduction.

1999. G. S. Weiss. *J. Geom. Anal.*

$$\frac{1}{r^d} F(u,B_r) - \frac{1}{r^{d+1}} \int_{\partial B_r} u^2 \nearrow$$

Non-existence of singular cones.

3D ← **2005.** L. Caffarelli, D. Jerison, C. Kenig.

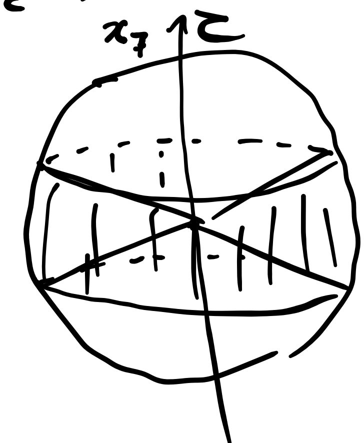
4D ← **2015.** D. Jerison, O. Savin. *Geom. Funct. Anal.*

the limits of

$$u_r(x) = \frac{1}{r} u(rx)$$

An example of singular cone.

7D **2009.** D. De Silva, D. Jerison. *J. Reine Angew. Math.*



Quantitative stratification.

2019. N. Edelen, M. Engelstein. *Trans. Amer. Math. Soc.*

Almost-minimizers.

2015. G. David, T. Toro. *Calc. Var. PDE*

2015. D. Bucur, D. Mazzoleni, A. Pratelli, B. Veichkov. *Arch. Rat. Mech. Anal.*

2018. L. Spolaor, B. Velichkov. *Comm. Pure Appl. Math.*

2019. G. David, M. Engelstein, T. Toro. *Adv. Math.*

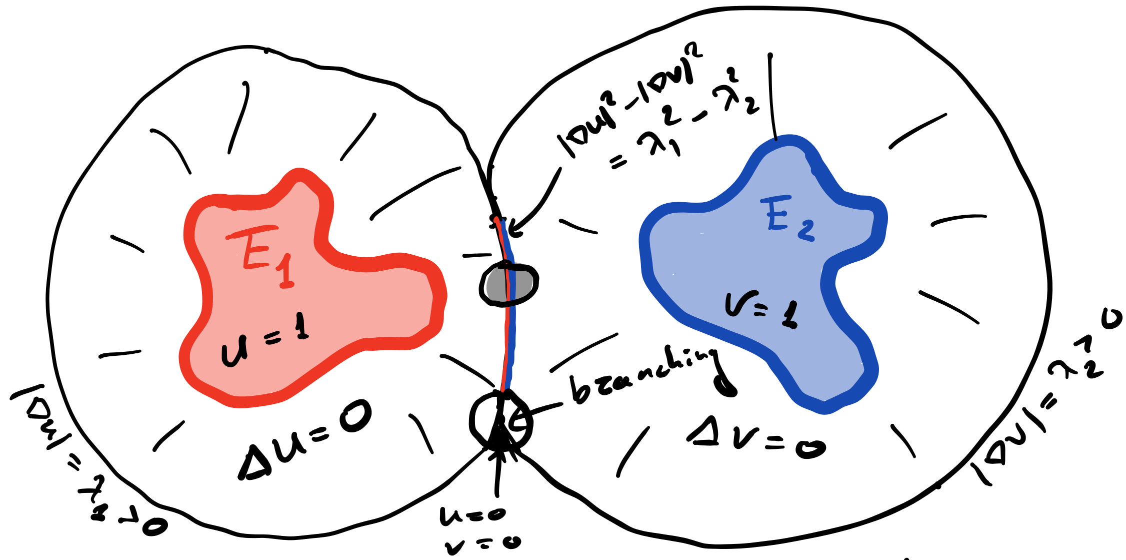
2020. L. Spolaor, B. Trey, B. Velichkov. *Comm. PDE.*

2020. D. De Silva, O. Savin. *Comm. PDE.*

$$\begin{cases} F(u,B_r) + Cr^{d+\alpha} \\ (1+r^{\alpha})F(u,B_r) + Cr^{d+\alpha} \end{cases}$$

II. The two-phase problem

E_1, E_2 - smooth disjoint



$u \geq 0, \ v \geq 0, \quad u \cdot v = 0$ on $\mathbb{R}^d$

Alt - Caffarelli - Friedman '84

$$\min \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 + |\nabla v|^2 + \lambda_1^2 |\{u>0\}| + \lambda_2^2 |\{v>0\}| \ : \ \begin{array}{l} u = 1 \text{ on } E_1 \ ; \ v = 1 \text{ on } E_2 \\ uv = 0 \ ; \ u,v \geq 0 \end{array} \right\}.$$

Two-phase free boundary ?

$\partial\{u>0\} \cap \partial\{v>0\}$

- 84' Alt-Caffarelli-Friedman : (2D

If in $|B_r \cap \{u=0\} \cap \{v=0\}| = 0$,

then $\partial\{u>0\} \cap \partial\{v>0\}$ is $C^{1,\alpha}$

2. The two-phase Bernoulli problem - biblio

2.1. Regularity of the interior two-phase boundary.

1984. H. W. Alt, L. A. Caffarelli, A. Friedman. *Trans. Amer. Math. Soc.* 2D

1987. L. Caffarelli. *Rev. Mat. Iberoamericana.*

1989. L. Caffarelli. *Comm. Pure Appl. Math..*

any dimension $D \geq 2$

2014. D. De Silva, F. Ferrari, S. Salsa. *Anal. PDE.*

2.2. Branching points.

2018. L. Spolaor, B. Velichkov. *Comm. Pure Appl. Math.*

2021. G. De Philippis, L. Spolaor, B. Velichkov. *Invent. Math.*

2D epiperimetric for minimizers

$D \geq 2$.

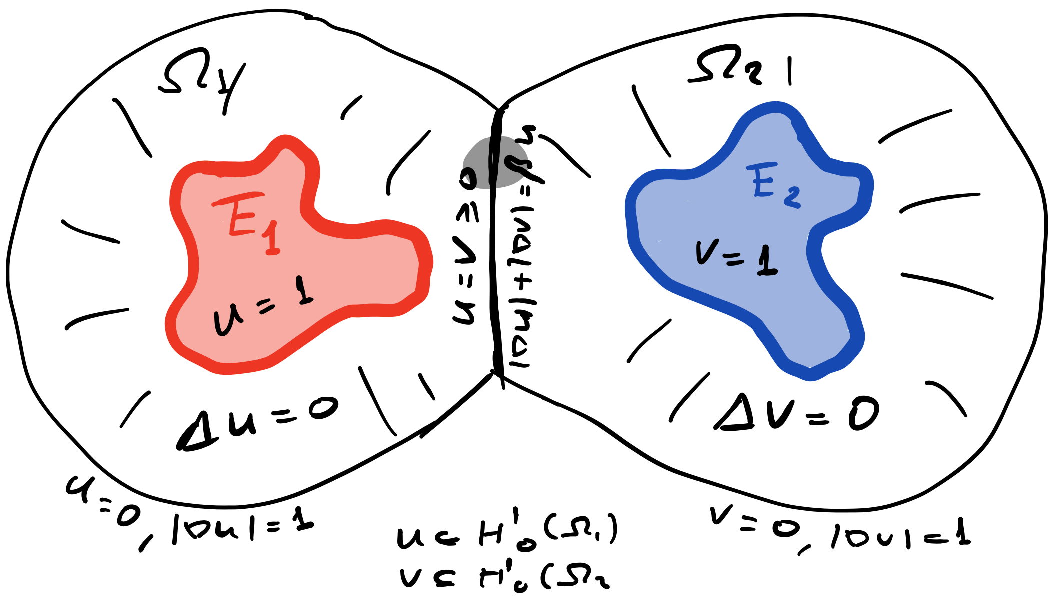
T: If $x_0 \in \partial\{u>0\} \cap \partial\{v>0\}$, then there is $r>0$ such that

$\partial\{u>0\}$ is $C^{1,\alpha}$ in $B_r(x_0)$

$\partial\{v>0\}$ is $C^{1,\alpha}$ in $B_r(x_0)$

you can take $\alpha = 1/2$.

III. Free boundary clusters

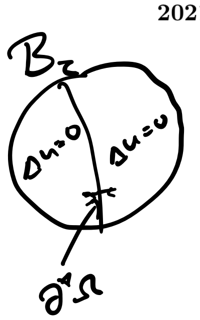


$$F = \int_{\Omega_1} |\nabla u|^2 + \int_{\Omega_2} |\nabla v|^2 + |\Omega_1| + |\Omega_2|$$

$$+ \beta \int_{\partial^*\Omega_1 \cap \partial^*\Omega_2} u^2$$

3.1. Interior regularity of the free interface.

2021. S. Guarino Lo Bianco, D.A. La Manna, B. Velichkov. *J. École Polytechnique.*



T: If (u, Ω) is a local minimizer of

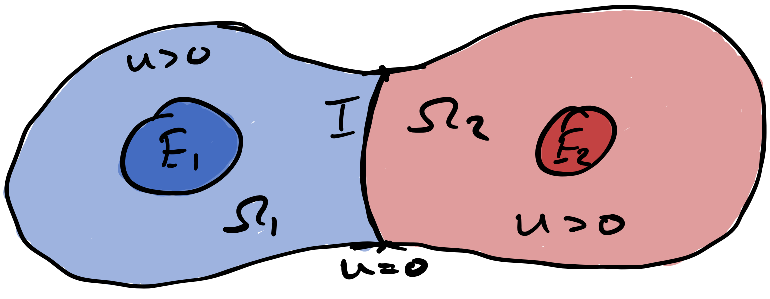
$$F(u,\Omega) = \int_{B_r} |\nabla u|^2 + \beta \int_{B_r \cap \partial^*\Omega} u^2$$

and $u > 0$ in B_r, then Ω is an almost-mi of the perimeter; Given E of finite perimeter in $\mathbb{R}^d$, a function $g \geq 0$, $g \in H^1(\mathbb{R}^d)$, $g > 0$ on B_r, then

$$\min \left\{ F(u,\Omega) \;:\; \begin{array}{l} u = g \text{ on } \partial B_r \\ \Omega = E \text{ on } \mathbb{R}^d \setminus B_r \\ u - g \in H^1_0(B_r) \\ \Omega \text{ - finite perimeter.} \end{array} \right\}$$

<u>Th</u>: In 2D, if E_1 and E_2 are smooth, then there is a solution to the pb:

$$\min \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 + |\{u>0\}| + \beta \int_{\partial^* \Omega_1 \cap \partial^* \Omega_2} u^2 \;:\; \begin{array}{l} u = 1 \text{ on } E_1 \text{ and } E_2 \\ \Omega_1 \supseteq E_1, \quad \Omega_2 \supseteq E_2 \\ \Omega_1 \cap \Omega_2 = \emptyset, \quad \{u>0\} \subseteq \Omega_1 \cup \Omega_2 \\ \Omega_1, \Omega_2 \text{ have finite perimeter in } \mathbb{R}^d. \end{array} \right\}$$



$\partial\{u>0\}$ is $C^{1,\alpha}$

I is $C^{1,\alpha}$ up to $\partial\{u>0\}$

3. Two-phase clusters - biblio

3.1. **Interior regularity of the free interface.**

2021. S. Guarino Lo Bianco, D.A. La Manna, B. Velichkov. *J. École Polytechnique.*

2021. L. A. Caffarelli, M. Soria-Carro, P. R. Stinga. *Arch. Rat. Mech. Anal.*

2021. H. Dong. *Ann. Appl. Math.*

3.2. **Regularity of the free boundary and the free interface.**

2022. S. Guarino Lo Bianco, D.A. La Manna, B. Velichkov.