The spherical Plateau problem

Antoine Song

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The hyperbolic part of M is

$$M_{hyp} := \bigcup_{j=1}^{k} H_j$$

(it is unique and has a unique hyperbolic metric g_{hyp}).

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Interpretations of the hyperbolic part M_{hyp}:

- M_{hyp} is the truly 3-dimensional part of M,
- given any Riemannian metric (M, g), the correctly normalized Ricci flow with surgery converges to (M_{hyp}, g_{hyp}) as $t \to \infty$,
- we'll see next that (M_{hyp}, g_{hyp}) is the unique solution of an infinite codimension Plateau problem.

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By Almgren (see also De Lellis-Spadaro), C_∞ is smooth outside a codimension 2 subset. Call such C_∞ a Plateau solution for $h \in H_n(N; \mathbb{Z})$.

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- White considered arbitrary complete normed abelian groups,
- Ambrosio-Kircheim considered any complete ambient metric space (subsequent works of Lang, Wenger, Schmidt, Sormani-Wenger...).

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De Pauw-Hardt developed a very general theory encompassing both directions.

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$$\lim_{i\to\infty} \text{mass}(C_i) = \inf\{\text{mass}(C); \quad C \in \mathscr{C}(h)\}.$$

Because $\dim(\mathcal{K}) = \infty$, C_i may not converge in any reasonable way to an integral current inside $\mathcal{K}$.

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given a sequence of boundaryless integral currents S_i of uniformly bounded masses and diameters, subsequentially there are a Banach space $\mathbf{Z}$, an integral current $S_\infty \subset \mathbf{Z}$, and isometric embeddings

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Rm: in fact here S_i converges in the intrinsic flat topology (Sormani-Wenger).

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Questions: Continuity of mass under $C_i \rightharpoonup C_\infty$? Regularity of C_∞?

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In general, C_∞ is only known to be an integral current. At least for an area-minimizing integral current T in a Hilbert space, Ambrosio-De Lellis-Schmidt showed that $\mathrm{spt}(T)$ is smooth in an open dense subset of $\mathrm{spt}(T)$. An Almgren type theorem seems plausible for such T.

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By $\pi_1(M)$-equivariance, it gives an embedding

$$B : (M, \frac{(n-1)^2}{4n} g_0) \to \mathcal{S}_{\partial\tilde{M}}/\pi_1(M).$$

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Questions: Uniqueness? What about non-locally symmetric manifolds?

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A Hilbert model for classifying spaces

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Fact: $\mathcal{S}^*(\Gamma)/\Gamma$ is a Hilbert manifold and a classifying space for Γ.

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Rm: in the special case where M is hyperbolic, $\mathcal{S}_{\partial \tilde{M}}/\pi_1(M)$ is not isometric to $\mathcal{K}$. In fact, M does not embed minimally in $\mathcal{K}$. Nevetherless $\mathcal{S}_{\partial \tilde{M}}/\pi_1(M)$ is isometrically embedded in the ultralimit of $\mathcal{K}$.

Hyperbolic manifolds and intrinsic uniqueness of Plateau solutions

Thm: Let (M, g_{hyp}) be a closed oriented hyperbolic manifold of dimension ≥ 3. Then any Plateau solution for h_M is intrinsically isometric to $(M, \frac{(n-1)^2}{4n} g_{hyp})$.

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Conjecture: For $n = 2$, any Plateau solution for h_M is intrinsically isometric to an element in the Deligne-Mumford compactification of $\{$hyperbolic metrics on $M\}$.

About the proof

Let $(M, \frac{(n-1)^2}{4n} g_{hyp})$, $\Gamma := \pi_1(M)$. Besson-Courtois-Gallot initiated the use of a "barycenter map" $\mathrm{bar} : \mathcal{S}_{\partial\tilde{M}}/\Gamma \to M$.

About the proof

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In our setting, we define a variant

$$\mathrm{bar} : \mathcal{K} \to M$$

such that $|\,\mathrm{Jac}_n \,\mathrm{bar}| \leq 1$ and when $|\,\mathrm{Jac}_n \,\mathrm{bar}|$ is close to 1, the differential $d\mathrm{bar}$ is almost an isometry.

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Steps :

1. construct a limit map $\mathrm{bar}_\infty : C_\infty \to (M, \frac{(n-1)^2}{4n} g_{hyp})$,
2. bar_∞ is volume preserving,
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Difficulty: lack of a priori regularity for C_∞: need to work on C_i and prove "almost" statements.

3-manifolds and intrinsic uniqueness of Plateau solutions

Thm: Let M be a closed oriented 3-manifold with hyperbolic part (M_{hyp}, g_{hyp}). Then any Plateau solution for h_M is intrinsically isometric to $(M_{hyp}, \frac{1}{3}g_{hyp})$.

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Question: Is the Bieberbach embedding stable under MCF?

A general structure result

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Thm: Any Plateau solution C_∞ for h embeds isometrically inside the spherical quotient $\mathbf{S}^\infty/\mathbf{\Gamma}_\infty$. Moreover the restriction of C_∞ to the smooth part of $\mathbf{S}^\infty/\mathbf{\Gamma}_\infty$ is mass-minimizing.

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Let $C_\infty^{>0} :=$ restriction of C_∞ to the smooth part of $\mathbf{S}^\infty/\mathbf{\Gamma}_\infty =$ noncollapsed part of C_∞.

The support of $C_\infty^{>0}$ is smooth on a dense open set by Ambrosio-De Lellis-Schmidt.

A general existence result

Combinatorial properties on $(\Gamma, h) \Longrightarrow$ Existence of non-trivial mass-minimizing integral currents of the form $C_{\infty}^{>0}$?

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Thm: Let Γ be a torsion-free hyperbolic group with $h \in H_n(\Gamma; \mathbb{Z}) \setminus \{0\}$ and $n \geq 2$. Then any Plateau solution C_∞ for h has a non-empty noncollapsed part $C_\infty^{>0}$.

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For instance, π_1 of negatively curved closed manifolds are torsion-free hyperbolic.