Space of convex ancient solutions to MCF

(work in progress with Angenent and Daskalopoulos)

- $F : M^n \hookrightarrow \mathbb{R}^{n+1}$

  $\frac{\partial}{\partial t} F = -H\nu$ (MCF)

- M^n is $\boxed{\text{compact}}$ $\Rightarrow$ finite time singularity T.

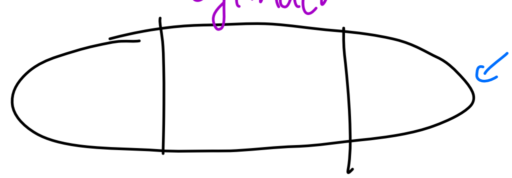
- Study finite time singularities through a blow-up procedure which yields as limits ancient solutions.

$\boxed{\text{def.}}$ Ancient solutions to MCF are solutions to (MCF) that exist for all $t \in (-\infty, a)$, $0 \leq a \leq +\infty$. If $a = +\infty$ we call them eternal solutions.

$\boxed{\text{Huisken:}}$ $M^n \hookrightarrow \mathbb{R}^{n+1}$ compact, convex hypersurface. The (MCF) starting at M^n has a smooth solution until finite time at which the flow extincts to a point. The rescaled MCF converges exponentially as $\tau \to +\infty$ to a round sphere S^n.

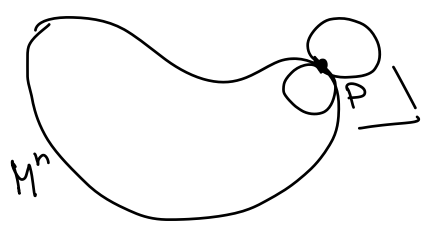
Examples of ancient solutions

- Solitons (self-similar): $S^k \times \mathbb{R}^{n-k}$, translating solitons (Bowl soliton)
- Examples that are not solitons are discovered by White; Haslhofer - Hershkovits. They have $O(j+1) \times O(n-j)$ symmetry, asymptotic to $S^j \times \mathbb{R}^{n-j}$ as $t \to -\infty$.



def. (Sheng, Wang): M^n is α-noncollapsed if

$\boxed{H>0}$



both of radius at least $\frac{\alpha}{H(p)}$, tangent to M^n at p.

Andrews: α-noncollapsed condition is preserved along the flow.

def. Ancient oval: compact, α-noncollapsed ancient MCF solution, not self-similar.

$\boxed{\text{Brendle, Choi:}}$ M_t, $t \in (-\infty, 0)$ be noncompact, strictly convex, uniformly 2-convex ($\lambda_1 + \lambda_2 \geq \beta H$) and noncollapsed ancient MCF solution in $\mathbb{R}^{n+1}$. It is the Bowl soliton.

$\boxed{\text{Angenent - Daskalopoulos - S.:}}$ Let M_t be uniformly 2-convex Ancient oval. Then, up to ambient isometries, translations and parabolic rescaling, it is the Ancient oval constructed by White; Haslhofer - Hershkovits.

$\boxed{\text{Choi, Haslhofer, Hershkovits, White:}}$ Obtained the classification from above assuming the parabolic blow down is a round cylinder. As a corollary they showed the flow through cylindrical and spherical singularities is unique.

. They also show the Mean convex neighborhood conjecture.

$\boxed{\text{Du, Haslhofer:}}$ Any nontrivial $\boxed{O(k)\times O(n+1-k)}$ symmetric ancient noncollapsed ancient MCF solution is up to scaling and time shift the one constructed by White; Haslhofer-Hershkovits.

- Tangent flow at time $-\infty$ is $\mathbb{R}^k \times S^{n-k}\left(\sqrt{2(n-k)|t|}\right)$.

- They also show $\forall\, k \geq 2$, there exists a $(k-1)$-parameter family of distinct Ancient ovals that are only $O(n+1-k)$ symmetric.

Question:

Facts about convex sets:

- $M = \partial \hat{M}$, $\hat{M}$ - convex, Then:

i) either $M = \mathbb{R}^n$

ii) or $M = \mathbb{R}^k \times N$, $0 \le k < n$ and $N = \partial \hat{N}$, $\hat{N}$ is closed convex set with interior that has no infinite line. N is homeomorphic to S^{n-k} or $\mathbb{R}^{n-k}$.

- If $M = \partial \hat{M}$ as above and M homeomorphic to $\mathbb{R}^n$, containing no infinite line, assume $\{x_{n+1} = 0\}$ is a supporting plane to $\hat{M}$ at the origin, then $D = \Pi(\hat{M})$ where $\Pi : \mathbb{R}^{n+1} \to \{x_{n+1} = 0\}$ is the standard orthogonal projection.

- D is called the $\boxed{\text{SHADOW}}$ of M.

- $\boxed{\text{Existence}}$ and $\boxed{\text{uniqueness}}$ of MCF starting at ∂C, where $C \subset \mathbb{R}^{n+1}$ is a convex set.

$\boxed{\text{Existence}}$: $C \subset \mathbb{R}^{n+1}$. Define its evolution by MCF to be

$$C_t = \overline{\bigcup D_t}$$

over all smooth compact solutions $\{D_s : 0 \leq s < S\}$ to MCF with $D_0 \subset \operatorname{Int} C_0$, $C_0 = C$.

- If C_0 is compact and has smooth boundary $\Rightarrow$ $C_t = C_t^{Hu}$.

Lemma: If $C_{k,t}$, $0 \leq t < T_k$ is a sequence of compact evolutions to MCF with $C_{k,0} \subset C_{k+1,0}$, then

a) $T_k \leq T_{k+1}$

b) $C_{k,t} \subset C_{k+1,t}$, $t \in [0, T_k)$

c) $C_{\infty,t} \overset{\text{def}}{:=} \bigcup_{k,\, t < T_k} C_{k,t}$ is the MCF evolution of $C_{\infty,0}$ and defined for $t < T_\infty$ with $T_\infty = \sup T_k$.

Regularity of MCF constructed above

Claim: ∂C_t is smooth for each $t > 0$ and $\{ \partial C_t \mid 0 < t < T \}$ is a smooth MCF solution.

$\boxed{\text{Uniqueness:}}$ Let $M_t^1, M_t^2 \subset \mathbb{R}^{n+1}$, $t \in (0,T]$ be two smooth convex solutions to MCF with

$\lim_{t \to 0} M_t^1 = \lim_{t \to 0} M_t^2 = M_0$. Then

$$\boxed{M_t^1 = M_t^2 \quad \forall\, t \in (0,T].}$$

proof:

Proof:

$\boxed{\text{Theorem:}}$ $\forall$ convex M_0, $\exists!$ smooth solution to MCF for $t \in [0, T]$.

- Want to define a notion of being α-noncollapsed for possibly noncompact, nonsmooth surfaces.

$\boxed{\text{def}}$ $M \subset \mathbb{R}^{n+1}$ complete, convex. We say ∂M is α-noncollapsed if $M = \bigcup_{k=1}^{\infty} C_k$, $\{C_k\}$ is an increasing sequence of smooth convex sets so that each ∂C_k is α_k-noncollapsed and the $\lim_{k\to\infty} \alpha_k = \alpha$.

$\boxed{\text{Lemma}}$ $C \subset \mathbb{R}^{n+1}$ complete, convex so that ∂C is α-noncollapsed. Let C_t be a unique smooth MCF coming out of C, $t \in [0,T)$. Then ∂C_t is also α-noncollapsed for $t \in (0,T)$.

- We put $\boxed{\text{weak topology}}$ on the space of convex sets

- $X = \{ C \subset \mathbb{R}^{n+1} \text{, convex with } \mathring{C} \neq \phi \}$

- $\forall\ C \in X$ define the Huisken measure

$$d\mu_C = (4\pi)^{-n/2} e^{-\frac{|x|^2}{4}} \underbrace{dH_C^n}_{\substack{\text{n-dim Hausdorff} \\ \text{measure on } \partial C}}$$

where

$$\mathcal{H}(C) = \frac{1}{(4\pi)^{n/2}} \int_{\partial C} e^{-\frac{|x|^2}{4}} dH_C^n$$

- $d\mu_C$ is given by : $\forall f \in C_c^0(\mathbb{R}^{n+1})$

$$\langle \mu_C, f \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{B_\varepsilon(C) \setminus C} e^{-|x|^2/4} f(x)\, dx$$

- Identify

$$X = \{ \mu_C \mid C \subset \mathbb{R}^{n+1} \text{ convex, } \mathring{C} \neq \phi \}$$

. If $\mathcal{C}$ is the space of bounded continuous functions $f:\mathbb{R}^{n+1} \to \mathbb{R}$, $\exists \lim_{\|x\|\to\infty} f(x)$, any measure μ on $\mathbb{R}^{n+1}$, defines a linear functional on $\mathcal{C}$ by

$$\langle \mu, f \rangle = \int f \, d\mu$$

. Regard X as a subspace of $\mathcal{C}^*$

. Topology on X is the weak* topology on $\mathcal{C}^*$.

$\boxed{\text{def}}$ $C_k \subset \mathbb{R}^{n+1}$ convex, $\mathring{C}_k \neq \emptyset$. We say $C_k \to C$ $(k\to\infty)$ in the strong sense if $d_{C_k}(x) \xrightarrow{k\to\infty} d_C(x)$, for all $x \in \mathbb{R}^{n+1}$, where $C_k, C \in X$ and $d_C(x) = d(x, C)$.

Lemma: If $\mu_{C_k} \to \mu_C$ as $k\to\infty$ in the weak sense, where $C_k, C \in X$, then $d(x, C_k) \xrightarrow{(k\to\infty)} d(x, C)$, $\forall x \in \mathbb{R}^{n+1}$.

- Define for $0 < h_0 < h_1 < 2$

$$X_\alpha(h_0,h_1) = \{ C \subset \mathbb{R}^{n+1} \mid \text{convex}, \ \mathring{C} \neq \emptyset, \ \partial C \text{ is } \alpha\text{-noncollapsed}, \ h_0 \leq \mathcal{H}(C) \leq h_1 \}$$

Lemma: $X_\alpha(h_0,h_1)$ is compact.

- Regard the RMCF

$$\boxed{\partial_t F = -H\nu + \tfrac{1}{2} F^\perp} \qquad (\text{RMCF})$$

as a flow on X.

- Fixed points of the flow: $\underbrace{S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}}_{\Sigma^k}$

$$1 = \mathcal{H}(\mathbb{R}^k) < \sqrt{2} < \cdots < \mathcal{H}(S^{k+1}) < \mathcal{H}(S^k) <$$
$$< \cdots < \mathcal{H}(S^1) < 2$$

- (RMCF) defines a map

$$\phi : D \to X$$

$$D = \{ (\mu_C, t) \in X \times [0,+\infty) \mid \mu_C \in X, \ 0 \leq t < T_C \}$$

$$\phi(\mu_C, t) = \phi^t(\mu_C)$$

- We show ϕ is a $\boxed{\text{continuous flow}}$ on $\mathcal{D}$.

- Define for $0 < h_0 < h_1 < 2$

$I(\alpha, h_0, h_1) = \{ C \in X \mid \partial C$ is α-non-collapsed, for which $\exists$ an entire solution $\{C_t\}_{t \in \mathbb{R}}$ of RMCF with $C_0 = C$ and $h_0 < \mathcal{H}(C_t) < h_1 \}$

$\boxed{\text{Proposition:}}$ For $0 < \alpha < \alpha_*$ and $0 < h_0 < h_1 < 2$, $h_i \neq \mathcal{H}(\Sigma^k)$ for all $0 \leq k \leq n$. Then

(i) $I(\alpha, h_0, h_1)$ is compact

(ii) $I(\alpha, h_0, h_1)$ consists of all fixed points Σ^k with $h_0 < \mathcal{H}(\Sigma^k) < h_1$, all hyperplanes through the origin and all connecting orbits between them.

$\boxed{\text{Conjecture:}}$ The set $I(\alpha, h_0, h_1)$ containing fixed points Σ^k is homeomorphic to an $(n-1)$-dimensional simplex.

- From now on we consider RMCF defined on

$$Z_s(\alpha) = \{ C \in Y_s(\alpha) \mid \tau(C) = 1 \}$$

$$Y_s(\alpha) = X_s(\alpha) / SO(n+1)$$

$$X_s(\alpha) = \{ C \in X \mid \partial C \text{ is } \alpha\text{-noncollapsed}, \; C = -C \text{ (invariant under point reflection)} \}$$

where $\tau(C)$ is the singular time for the (MCF).

$\boxed{\text{Remark:}}$ $C \in X_s(\alpha)$ is either compact or is of the form $\tilde{C} \times \mathbb{R}^k$ for some compact symmetric space $\tilde{C}$.

- $I_s(\alpha, h_0, h_1) = I(\alpha, h_0, h_1) \cap X_s(\alpha)$

- $\boxed{\text{In } \mathbb{R}^3}$:

unique orbit by our result {ADS}

$$S^1 \times \mathbb{R} \longrightarrow S^2$$

- $\boxed{\text{In } \mathbb{R}^4}$

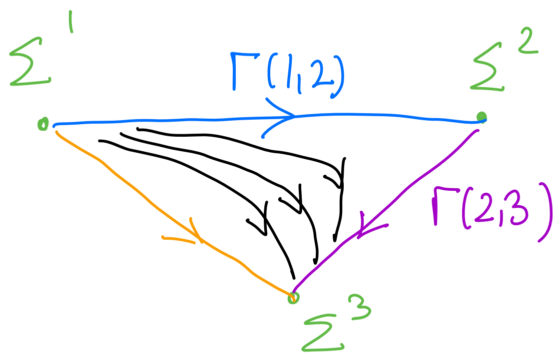
$$\Sigma^1 = S^1_{\sqrt{2}} \times \mathbb{R}^2, \quad \Sigma^2 = S^2_{\sqrt{4}} \times \mathbb{R}, \quad \Sigma^3 = S^3_{\sqrt{6}}$$

- There are connecting orbits from Σ^i to Σ^j whenever $i<j$.

- {ADS}: The orbits from Σ^1 to Σ^2 and from Σ^2 to Σ^3 are unique:

$\boxed{\text{Du-Haslhofer}}$ (a) There is 1-parameter family of orbits $\Sigma^1 \to \Sigma^3$ that are $O(2)$-symmetric

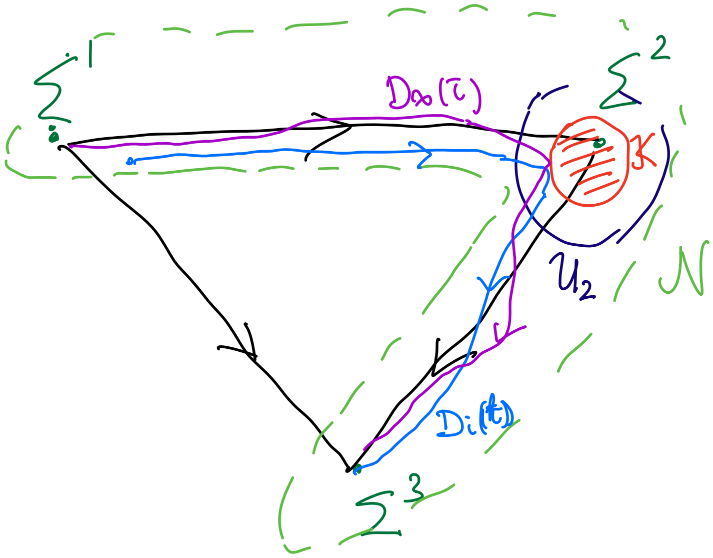
(b) There is an orbit which is $O(2)\times O(2)$ symmetric and it is unique



in $\mathbb{R}^4$

$\boxed{\text{Theorem:}}$ Let $N \subset Z_s(\alpha)$ be any open neighborhood of $\Gamma(1,2) \cup \Gamma(2,3)$. Then N contains a connecting orbit from Σ^1 to Σ^3.

Proof:



- $\mathcal{U}_2$ is an open set in $Zs(\alpha)$
- $K \subset \mathcal{U}_2$ is compact

Step 1: