Min-max minimal hypersurfaces with multiplicity two

joint with Zhichao Wang.

- Volume spectrum / min-max theory.

$\mathcal{Z}_n(M^{n+1}, g, \mathbb{Z}_2)$ = space of mod-2 hypercycles.

$$\Big[\quad \mathcal{C}(M^{n+1}) = \{ \Omega \subset M^{n+1},\ |\partial\Omega| \ (= \text{Area}(\partial\Omega)) < +\infty \}$$

$$\mathbb{Z}_2 \curvearrowright \mathcal{C}(M^{n+1}),\quad \Omega \to M \setminus \Omega$$

$$\mathcal{C}(M^{n+1}) / \mathbb{Z}_2 =: \mathcal{Z}_n(M^{n+1}, g, \mathbb{Z}_2) \quad \Big]$$

$$\cong \mathbb{RP}^\infty \quad (\text{Almgren } 61).$$

k-sweepouts $:\Rightarrow \Phi: X \longrightarrow \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ e.g. $X = \mathbb{RP}^k$

if $\quad \Phi^*(\bar\lambda^k) \neq 0$ in $H^k(X, \mathbb{Z}_2)$

$$\Big[\quad H^*(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\bar\lambda]$$

$$\bar\lambda \in H^1(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2)$$

k-th volume spectrum (Gromov, Guth, Marques-Neves)

$$\omega_k(M^{n+1}, g) = \inf_{\Phi: X \to \mathcal{Z}_n \ k\text{-sw.}} \ \max_{x \in X} |\Phi(x)|$$

- $0 \overset{\text{Almgren}}{<} \omega_1 \leq \omega_2 \leq \cdots \omega_k \longrightarrow +\infty \sim k^{\frac{1}{n+1}}$

<u>Min-max Thm</u> (Almgren-Pitts . Schoen-Simon, Marques-Neves)

$3 \leq n+1 \leq 7$, $\forall k \in \mathbb{N}$.

$$\omega_k(M,g) = \sum_{i=1}^{l_k} m_i^k \, |\Sigma_i^k|$$

$m_i^k \in \mathbb{N}_{>0}$, Σ_i^k: C^∞ min hypersurf.

$$\sum_{i=1}^{l_k} \mathrm{ind}(\Sigma_i^k) \leq k .$$

<u>Multiplicity</u>:

- $k=1$, $Ric_g > 0$: Marques-Neves 11. Z. 13. M-N-Ketover 15

  Σ^1 is 2-sided, $m^1 = 1$.

- $n+1 = 3$ generic g : Chodosh - Mantoulidis 18

  Σ_i^k 2-sided . $m_i^k = 1$

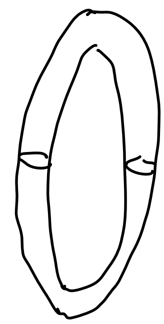
- $3 \leq n+1 \leq 7$ generic g. Z. 19.

  Σ_i^k 2-sided . $m_i^k = 1$

<u>Cor</u>: g is non-bumpy . if Σ_i^k is 1-sided or $m_i^k > 1$ $\Rightarrow$ Σ_i^k is weakly stable

<u>Question</u>: whether $m_i^k > 1$ (or Σ_i^k 1-sided) can happen ?

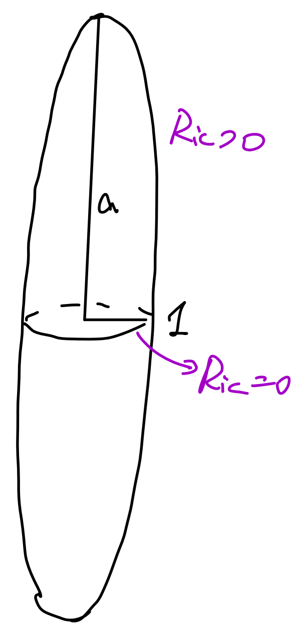
Trivial Example: $M^{n+1} = S^n(1) \times S^1(a) \quad a >> 1$



$$\omega_1(M, g) = 2\,|S^n(1) \times \{t\}|$$

$$= |S^n(1) \times \{0\}| + |S^n(1) \times \{\varepsilon\}|.$$

<u>Thm</u> (Wang - Z. 22). $\underline{S^{n+1}_a} \subseteq \mathbb{R}^{n+2} \qquad 3 \le n+1 \le 7.$



$$x_1^2 + \cdots + x_{n+1}^2 + \underline{\frac{x_{n+2}^{2n}}{a^{2n}}} = 1$$

$a >> 1$

$$S^n_0 = S^{n+1}_a \cap \{x_{n+2} = 0\}$$

$$\omega_2(S^{n+1}_a) = 2\,|S^n_0|.$$

1-sweepouts 2-sweepouts.

- $f: M^{n+1} \longrightarrow \mathbb{R}$ Morse.
- $\Phi_1: \mathbb{RP}^1 \xrightarrow{= \mathbb{R}^2 \setminus \{0\} / \mathbb{R}^*} \mathcal{Z}_n(M, \mathbb{Z}_2)$.
  $[a, b] \longrightarrow [\,\partial\{af(x) + b > 0\}\,]$
- $\Phi_2: \mathbb{RP}^2 \longrightarrow \mathcal{Z}_n$
  $[a, b, c] \longrightarrow [\partial\{af^2(x) + bf(x) + c > 0\}]$.

(*): Φ_2 picks up 2 slices of Φ_1

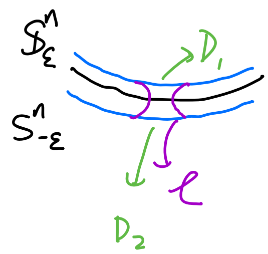
$M = S^{n+1}_a$: • Φ_1 is an optimal 1-s.w. $\quad (a > n)$.

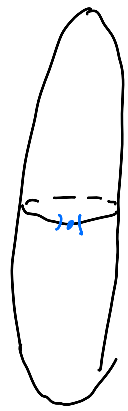
$$\omega_1(S^{n+1}_a) = \max_{x\in[-a,a]} |\Phi_1(x)| = |S^n_0|$$

- $$\omega_2(S^{n+1}_a) \le \max_{x\in\mathbb{RP}^2} |\Phi_2(x)| \le 2|S^n_0| .$$

Catenoid- Estimates   M-N-K.   Halshofer-K.

$$x_1^2+\cdots+x_{n+1}^2+\frac{x_{n+2}^2}{a^2}=1$$





$$|S^n_\varepsilon|+|S^n_{-\varepsilon}|-|D_1|-|D_2|+|e|$$
$$< 2|S^n_0|$$

*): on S^{n+1}_a, the converse holds!

Outline of Proof:

Suppose $\omega_2(S^{n+1}_a, g) = |\Sigma_a|$ with multiplicity $=1$ $\Sigma_a \neq S^n_0$.

Lemma: $a\to+\infty$, $\Sigma_a \to 2S^n_0$.

- by M.P. $\Sigma_a\cap S^n_0\neq\emptyset$.
- $|\Sigma_a| = \omega_2 \le 2|S^n_0|$ & $\mathrm{ind}(\Sigma_a)\le 2$.
- $S^{n+1}_a \longrightarrow S^n(1)\times\mathbb{R}$.

$$\omega_2(S^{n+1}_a)\longrightarrow\omega_2(S^n(1)\times\mathbb{R}) = 2|S^n_0| .$$

Thm A: $|\Sigma_a| > 2|\mathbb{S}^n_0|$. "$\Rightarrow \neq$".

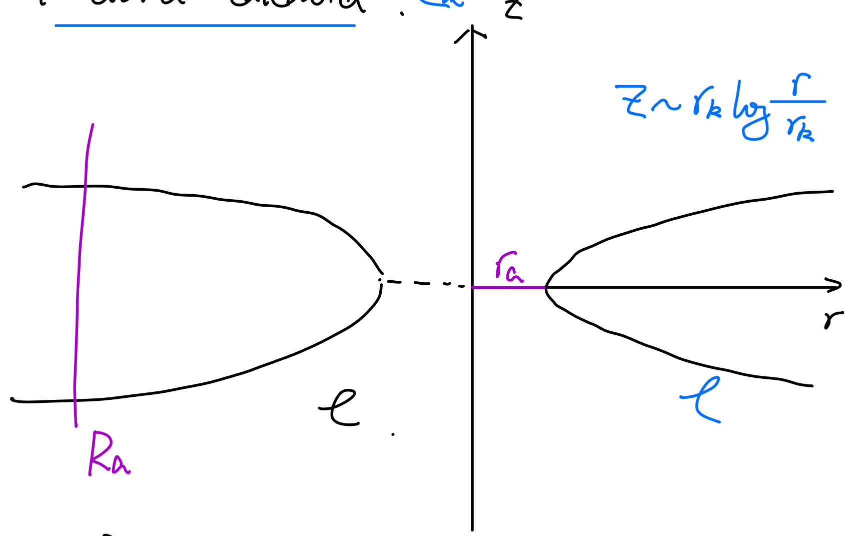
- Area-height estimates.
- "$n+1 = 3$".
- $\Sigma_a \to 2\,\mathbb{S}^n_0$ as $a \to +\infty$. "has 1 neck"
- Blow up analysis at the singular pt p.

  $\longrightarrow$ a standard Catenoid $\mathcal{C}_a$.

$|\mathcal{C} \cap B_{R_a}|$

$\hat{=}\ 2 \cdot 2\pi \cdot R_a^2$

$+\ 2\pi(\log R_a - 1)\, r_a^2$



$R_a \to 0$

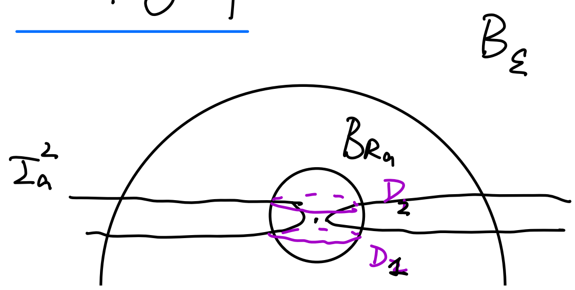
$R_a / r_k \to +\infty$

- Remaining part.

Thm B: $\mathrm{dist}_H(\Sigma^i_a, \mathbb{S}^n_0) \le C \cdot r_a |\log r_a|$

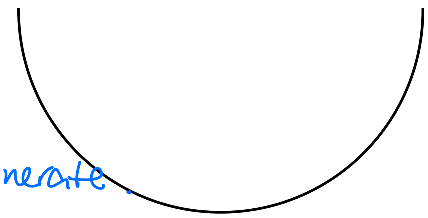
Pf of Thm A:

$|\Sigma^2_a \setminus B_{R_a}| + |D_2|$



$$\geq |S_0^2| - C\,(r_a|\log r_a|)^3$$

$\downarrow$ degenerate



$$|\Sigma_a^1 \setminus B_{R_a}| + |\Sigma_a^2 \setminus B_{R_a}| \geq 2|S_0^2| - 2\cdot 2\pi\cdot R_a^2 - C|r_a \log r_a|^3$$

$+$

$$|\mathcal{C} \cap B_{R_a}| \geq 2\cdot 2\pi R_a^2 + C\cdot r_a^3$$

$r_a^2 \cdot r_a|\log r_a|^3 \downarrow 0$ as $r_a \to 0$

$//$

$$|\Sigma_a| > 2|S_0^2|.$$

- w_a: height between Σ_a^1 & Σ_a^2

① $$\Delta w_a \leq w_a + \frac{w_a^3(x)}{dist(x,p)^4}$$

$$\operatorname{div}\left(\frac{\nabla w_a}{\sqrt{|\nabla w|^2+1}}\right) = 0$$

w_a = scaled down of $\mathcal{C}$.

$$|\nabla w_a| \sim \frac{w_a}{d(x,p)} \qquad |\nabla^2 w_a| \sim \frac{w_a}{d^2(x,p)}$$

② $$I_a(s) = \fint_{\partial B(p,s)} w_a\, |\nabla^t dist(\cdot,p)|$$

$\tau_a(s)$

$F_a(s)$