Regularity results for the special Lagrangian equation

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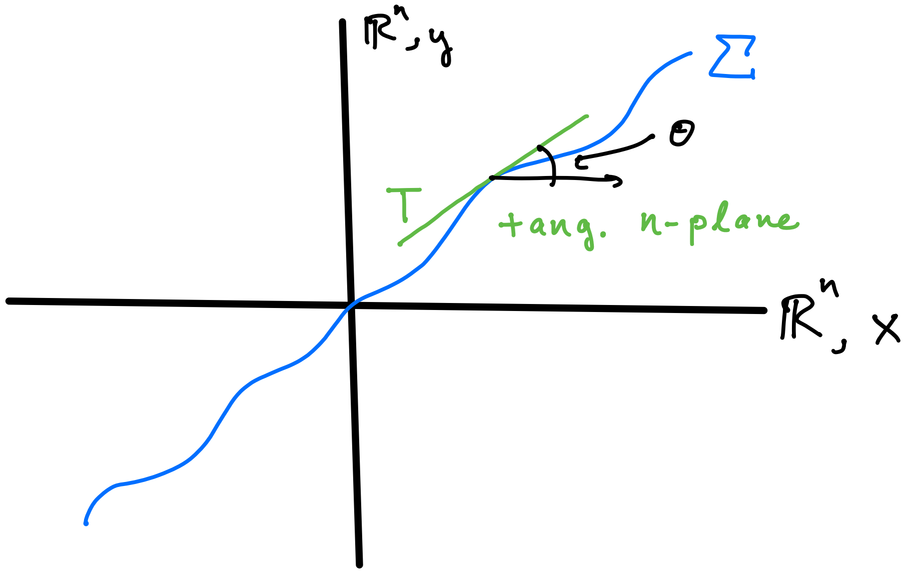
Let $F : \mathbb{R}^n \to \mathbb{R}^n$, $\Sigma = \text{gr.}(F)$



- Σ Lagrangian if $JT \perp T$, $J(x,y) = (-y,x)$

  $\iff F = \nabla u$, $u : \mathbb{R}^n \to \mathbb{R}$

- induced metric:

$$g = I + (D^2u)^2 = \begin{pmatrix} 1+\lambda_1^2 & & \\ & \ddots & \\ & & 1+\lambda_n^2 \end{pmatrix}$$

eigvals (D^2u)

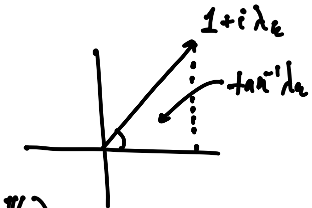
• Lag. angle: $\dfrac{\det(I + iD^2u)}{\sqrt{\det g}} \overset{(\star)}{=} e^{i\theta(x)}$

$$\vec{H} = J\,\underline{\nabla_\Sigma}\,\theta \rightsquigarrow \underbrace{\Sigma \text{ minimal if } \theta = \text{const}}_{\text{"special Lagrangian"}}$$

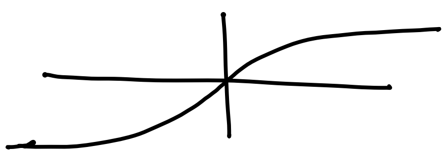
in fact, Σ is vol. minimizing if $\theta = \text{const}$:

$\mathrm{Re}\,(e^{-i\theta}\,dz_1 \wedge \dots \wedge dz_n)$ calibrates $(z_k = x_k + iy_k)$

• Other forms of $\star$:



(1) $F(D^2u) := \underset{\xi}{\sum_{k=1}^{n} \tan^{-1}(\lambda_k)} = \theta$, $\theta \in (-n\pi/2, n\pi/2)$



F degenerate elliptic

locally unif. elliptic

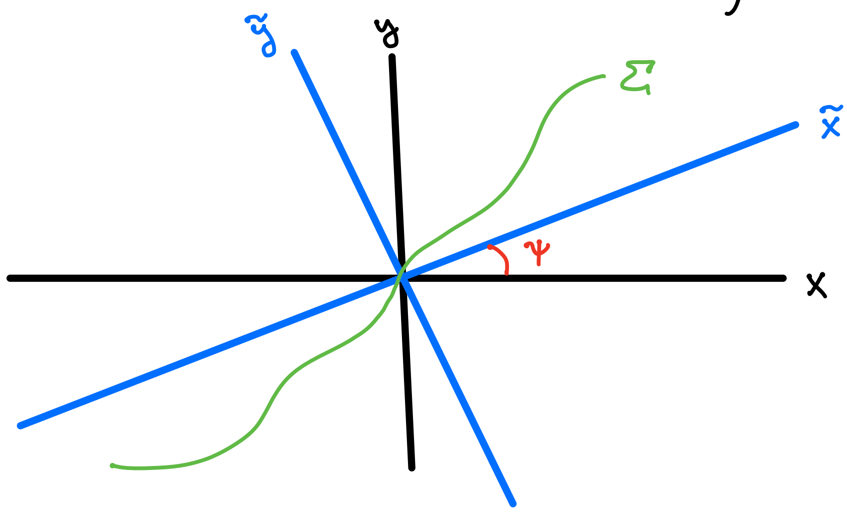
(2) $(1 - \sigma_2 + \dots) + i(\sigma_1 - \sigma_3 + \dots) = \sqrt{\det g}\cdot(\cos\theta + i\sin\theta)$

$\underline{n=2}$: $\theta = 0 \rightsquigarrow \Delta$ eqn, $\theta = \pi/2$, M-A eqn

$\underline{n=3}$: $\theta = 0 \rightsquigarrow \det = \Delta$, $\theta = \pi/2$, $\sigma_2 = 1$.

<u>Rmk</u>: can continuously pass through phases by rotating Σ:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



in new coords, $\theta \longmapsto \theta - n\psi$.

If Σ still graphical, $= gr.(\nabla\tilde{u})$,

$$F(D^2\tilde{u}) = \theta - n\psi.$$

e.g. $\pi/2$-rot. $= -$Legendre transform (swap x, ∇u)

or $n=2$, $\pi/4$-rot takes MA $\longleftrightarrow$ Δ eqn.

"$\underbrace{\text{Legendre – Lewy transform}}_{\text{useful when } u \text{ sat. convexity properties}}$" in papers of Yuan et al.

useful when u sat. convexity properties

↓

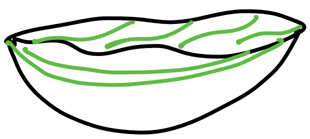
rot. grad. graph still graphical.

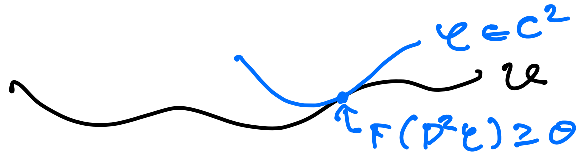
PDE, known results ($\theta = \underline{\text{const}}$.)

1.) $\exists !$ visc. sol'n $u \in C(\overline{B_1})$ to

$$\begin{cases} F(D^2 u) = \sum_{k=1}^{n} \tan^{-1}(\lambda_k) = \Theta, \\ u|_{\partial B_1} = \varphi \in C(\partial B_1). \end{cases}$$

(Perron's Method; Ishii, late 1980s)





2.) a) $\varphi \in C^\infty$ + $|\Theta| \geq (n-2)\pi/2 \Rightarrow u \in C^\infty(\overline{B_1})$, C-N-S '85

b) $|\Theta| \geq (n-2)\pi/2 \Rightarrow u \in C^\infty(B_1) \cap C(\overline{B_1})$, Wang-Yuan '14

(Key: eqn $\underline{\text{concave}}$; $\Delta_g \lambda_{max} \geq 0$)

a: Boundary C^2 est + Evans-Krylov

b: A priori Int. C^2 est ("Bombieri-De G.-Miranda for ∇u" solves MSS)

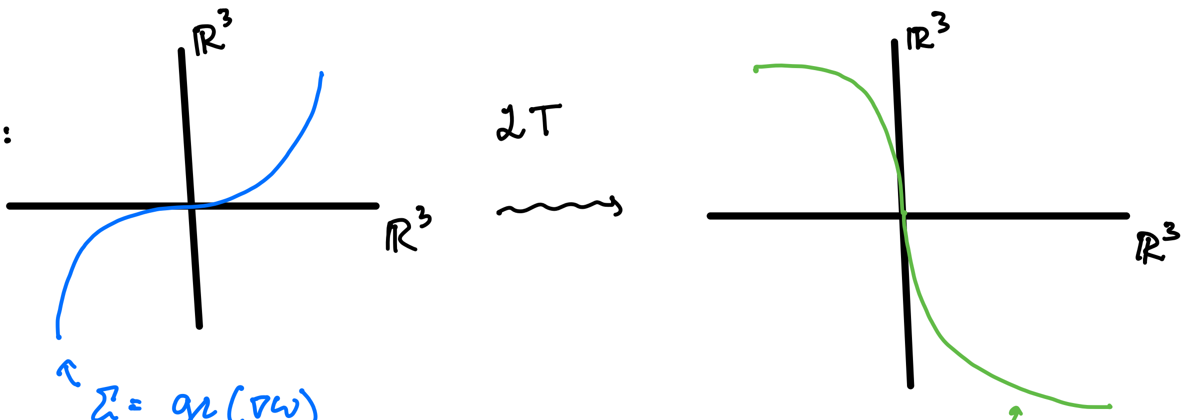
+ Evans-Krylov, 2a).

3.) $\exists$ sing. sol'ns $\in C^1 \setminus C^2$ $(B_1 \subset \mathbb{R}^n)$, $|\Theta| < (n-2)\pi/2$

- ex's $\in C^{1,1/3}$ by Nadirashvili-Vladuts, '10

• ex's $\in C^{1,\frac{1}{2m-1}}$ any $m \geq 2$, by Wang-Yuan, '13

idea:



$\Sigma = gr(\nabla w)$

$\sigma_2(D^2 w) = 1 \quad (\theta = \pi/2)$

$w \approx 1/2\,(x^2+y^2) - z^{2m} + h.o.t.$

$\tilde{\Sigma} = \text{graph}(\nabla u)$

$u = -w^* \approx -\frac{1}{2}(x^2+y^2) + z^{\frac{2m}{2m-1}}$

$\sigma_3(D^2 u) = \sigma_1(D^2 u) \quad (\theta = 0)$

Open questions:

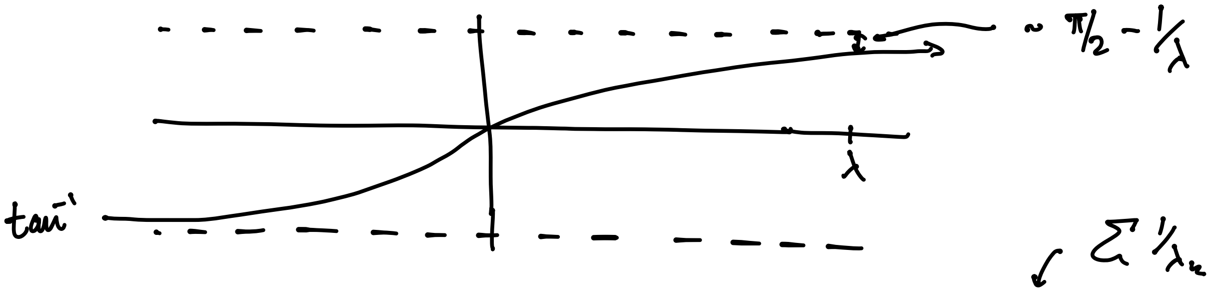
(1) $\|\nabla u\|_{L^\infty(B_{1/2})} \overset{?}{\leq} C(n, \|u\|_{L^\infty(B_1)})$, $|\theta| < (n-2)\pi/2$ const. or $\theta \in C^1$?

(2) Do $\exists$ sing. sol'ns w/ non-analytic ∇ graph, (e.g. non-flat graphical sLag cones) ?

Modest progress on (1):

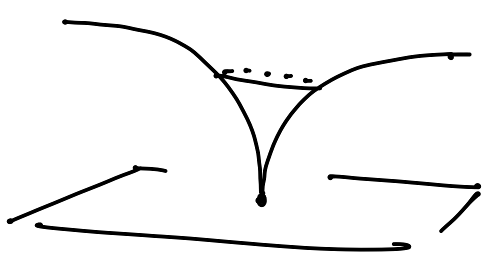
natural counterexample candidates satisfy

$$\begin{cases} \text{(a) } u \ \gamma\text{-homogeneous}, & 0<\gamma<1 \\ \text{(b) } |\det D^2u| > 0 \text{ on } S^{n-1} \end{cases}$$



$$\Rightarrow F(D^2u) = \underbrace{\text{const.} - |x|^{2-\gamma}\left(\overbrace{\frac{\sigma_{n-1}(D^2u)}{\det D^2u}}^{\sum 1/\lambda_i}\Bigg|_{S^{n-1}}\right)}_{\in C^1.} + \text{h.o.t.}$$

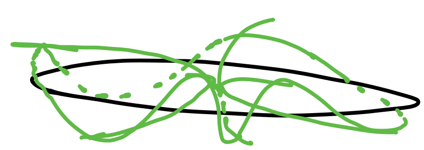
e.g. $|x|^\gamma$



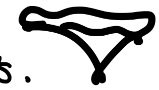
const. $= (n-2)\pi/2$

<u>issue</u>: this ex not visc. sol'n @ 0. Need

(c) u changes sign.



Thm A (M.,'22). There are no fcns on $\mathbb{R}^n$ satisfying (a), (b) and (c) simultaneously.

· $\rightsquigarrow$ natural shapes of c-ex's are ruled out. 

· rmk: $r^\alpha \cos(k\theta)$ satisfy (b), (c), $\begin{cases} \alpha > 1 \text{ or } < 0 \\ k(\alpha) \text{ large}. \end{cases}$

· Follows from:

Thm B. (M., '22)

Let $\Omega =$ cone over $\Gamma \subset S^{n-1}$. If $\exists\, u$ s.t.

$$\begin{cases} (a) & u \ \alpha\text{-homog. in } \Omega \quad (\alpha \in (0,1)) \\ (b) & |\det D^2 u| > 0 \text{ on } \Gamma \\ (\tilde{c}) & u > 0 \text{ in } \Omega, \ u|_{\partial\Omega} = 0 \end{cases}$$

$\Rightarrow \Omega^c$ is a cvx cone + hemisphere $\subset \Gamma$,

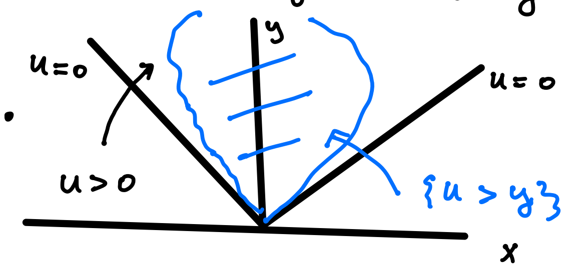
+ $D^2 u$ has exactly 1 neg. eigenvalue in Ω.

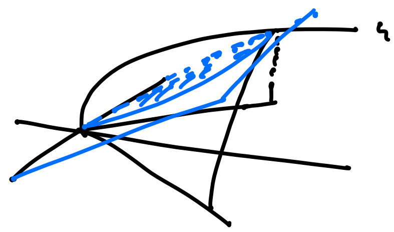
· (b) can be relaxed to $\det D^2 u > 0$ or < 0, $K := \frac{|D^2 u|^n}{\det D^2 u} \in L^p_{loc}(\Omega)$, $p > n-2$

· "Function-theoretic" result, related also to mappings w/ bnd'd dilatation (elasticity theory) + M-A eqns w/o convexity hypotheses on sol'n.

Proofs of thm. A + B.

I. Th. A, $n=2$, topological proof

- $\alpha<1$ + sgn-changing $\Rightarrow \det D^2u<0$ on S^1.
- 



+ max principle.

$n\geq 3$: topology / geom. of cones is complicated... need a new approach.

II. Th. B, $n=2$, analytical proof.

Let $v=\begin{cases} u^{1/\alpha}, & \Omega \\ 0, & \text{otherwise}\end{cases}$

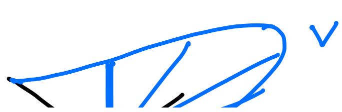
$$\Rightarrow \begin{cases} v \ \text{1-homogeneous}, \\ \underbrace{v=|\nabla v|}_{\text{uses } \alpha<1}=0 \ \text{on } \partial\Omega. \end{cases} \quad *$$

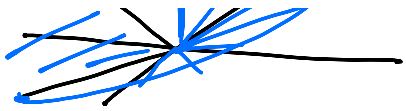
Calculation: $\underbrace{\Delta v = C_\alpha u^{\frac{1}{\alpha}-2}\det D^2u}$ on Γ

+ $*$ $\Rightarrow$ v is <u>convex</u>

$\Rightarrow \{v=0\}=\Omega^c$ is a cvx cone.

∎.





III. Th. B, pf in general dimension.

v: same as above (1-homog, $\in C^1(\mathbb{R}^n\setminus\{0\})$).

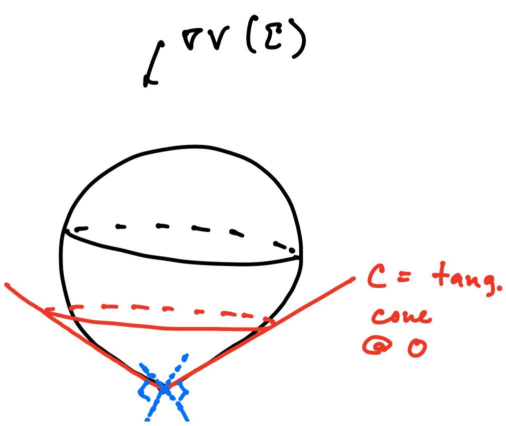
Calculation: $\boxed{\sigma_{n-1}(D^2 v) = C_{\gamma,n}\, u^{\frac{n-1}{\gamma}-n} \det D^2 u}$ on Γ

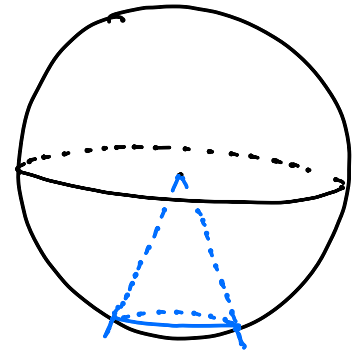
study geom$(\nabla v(\Gamma))$:

calculus: $\mathrm{II}|_{\nabla v(x)} = (D^2 v)^{-1}(x)$.

Touch $\nabla v(\Gamma)$ by a sphere : v crx $\rightsquigarrow$ Ω^c crx. ☒

in fact: $v =$ supp fcn $(\nabla v(\Gamma))$:





Ω^c

Ω^c = $-$cvx dual of C.

Thank you for your attention!