Boundary regularity of area-minimizing currents: a linear model with analytic interface

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MSRI Hot Topics Workshop:
Regularity Theory for Minimal Surfaces and Mean Curvature Flow

Background

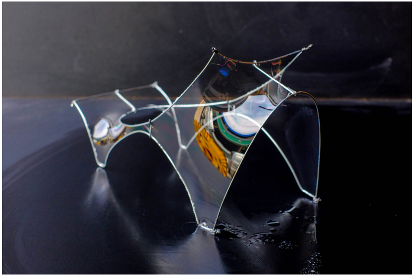

Figure: Soap film. Photo credit: archdaily.com

Plateau's problem: Given a closed curve Γ, what is the surface T that spans Γ with the least area?

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Depending on what class of surfaces we are minimizing, there are **different formulations of Plateau's problem**:

- Classical area-minimizing surfaces using the parametric approach (as images of a disk, or surfaces of higher genus Σ_g), *Douglas, Radó, Courant et al.*
- Integral currents, *Federer-Fleming et al.*
- Set-theoretic approach, *Reifenberg, Harrison-Pugh, David et al.*
- Integral varifolds, *Almgren, Allard et al.*

Integral currents: model orientable submanifolds

In 1960 Federer and Fleming introduced the notion of *integral current* and proved the existence of area-minimizers in this class.

Definition

We say a current $T \subset \mathbb{R}^{m+n}$ of dimension m is **integer rectifiable**, if there are

- countably m-dimensional *orientable* C^1 *submanifolds* $M_i \subset \mathbb{R}^{m+n}$,
- pairwise disjoint closed sets $A_i \subset M_i$,
- positive integers $k_i \in \mathbb{N}$,

such that

$$T = \sum_i k_i [\![A_i]\!], \quad \text{modulo a set of zero } \mathcal{H}^m\text{-measure.}$$

We say a current T is **integral** if both T and its boundary ∂T are integer rectifiable.

Area-minimizing current

Definition (Area-minimizing current)

Let T be an m-dimensional integral current in $\mathbb{R}^{m+n}$ (or in a Riemannian manifold M^{m+n}). We say T is *area minimizing*, if

$$\text{Area}(T') \geq \text{Area}(T) \text{ for any } \textit{competitor } T' \text{ of } T,$$

that is, T' is an integral current such that $T' = T$ outside of some compact set.

Interior regularity: codimension one ($n = 1$)

Theorem (De Giorgi, Simons, Federer, Simon et al.)

Assume T is an area-minimizing current and $n = 1$. Then

1. *If $2 \leq m \leq 6$, T is regular, i.e. $\mathrm{Sing}_i(T) = \emptyset$.*
2. *If $m = 7$, $\mathrm{Sing}_i(T)$ consists of isolated points.*
3. *If $m > 7$, $\mathrm{Sing}_i(T)$ has Hausdorff dimension at most $m - 7$, and it is $(m - 7)$-rectifiable.*

Interior regularity: higher codimension ($n \geq 2$)

When the codimension $n \geq 2$, *branch point* starts to emerge.

Example of branch singularity. The holomorphic curve

$$C := \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$$

is a 2-dimensional area-minimizing current in $\mathbb{R}^4$. The origin is singular despite that C has a flat tangent plane $\{(z, w) \in \mathbb{C}^2 : z = 0\}$ at the origin.

Interior regularity: higher codimension ($n \geq 2$)

Theorem (Almgren, Chang, DeLellis-Spadaro, DeLellis-Spadaro-Spolaor)

Assume T is an area-minimizing current and $n \geq 2$. Then

1. *When $m = 2$, $\mathrm{Sing}_i(T)$ consists of isolated points.*
2. *$\mathrm{Sing}_i(T)$ has Hausdorff dimension at most $m - 2$.*

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- In particular, if T is a two-dimensional area-minimizing current, then locally $\mathrm{spt}(T)$ is a branched minimal surface.
- By a dimension reduction argument, it suffices to study *flat* singular points.

Almgren's proof of interior regularity

1. Near a flat singular point p, approximate the current T locally by the graph of a multi-valued function

$$f : \mathbb{D} \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$$

where $\mathcal{A}_Q(\mathbb{R}^n)$ is the metric space of unordered Q-tuples of points in $\mathbb{R}^n$, and the integer Q equals $\Theta(T, p) := \lim_{s\to 0} \frac{\|T\|(\boldsymbol{B}_s(p))}{\omega_m s^m}$.

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   The reason is that
$$\|T\|(\boldsymbol{C}_r(x)) - Q \cdot \omega_m r^m \approx \frac{1}{2} \int_{B_r(x)} |Df|^2,$$
   where $\boldsymbol{C}_r(x)$ denotes the cylinder $B_r(x) \times \mathbb{R}^n \subset \mathbb{D} \times \mathbb{R}^n$.

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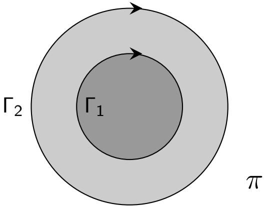
3. Construct the *central manifold*, to make sure singularity does not disappear after the blow-up.
4. Prove analogous regularity result for multi-valued functions which minimize the Dirichlet energy.

Boundary regularity

On the boundary, a regular point can be either *one-sided* or *two-sided*.

Example. Let π be a two-dimensional plane and $\Gamma = \Gamma_1 \cup \Gamma_2$. The area-minimizing current T which bounds Γ is the sum of the two disks bounded by Γ_1 and Γ_2, counting multiplicity.



The first boundary regularity result is by Allard (for varifolds):

Theorem (Allard 1969)

1. *If* $p \in \Gamma$ *is a point where the density* $\Theta(T, p)$ *equals* $\frac{1}{2}$*, i.e.* p *is one-sided, then* $p \in \mathrm{Reg}_b(T)$*.*
2. *If there is some wedge* W *of opening angle smaller than* π *whose tip contains* p *and such that* $\mathrm{spt}(T) \subset W$*, then* $\Theta(T, p) = \frac{1}{2}$ *and thus* $p \in \mathrm{Reg}_b(T)$*.*

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Remark

DeLellis-Nardulli-Steinbrüchel: When $\partial T = Q[\![\Gamma]\!]$, any boundary point $p \in \Gamma$ with density $< \frac{Q+1}{2}$ is regular.

Boundary regularity in codimension one

Theorem (Hardt-Simon 1979)

Let $\Gamma \subset \mathbb{R}^{m+1}$ be a $C^{1,\alpha}$ closed oriented embedded submanifold of dimension $m-1$. Suppose T is an area-minimizing current with boundary Γ, then $\mathrm{Sing}_b(T) = \emptyset$.

Remark

In particular when $m = 2$, $\mathrm{spt}(T)$ is an embedded surface (with boundary) of finite genus.

Boundary regularity in higher codimensions

Again, the case of higher codimensions is different.

- **Genuine branch singularity**. For example, cut the minimizing current $\{(z, w) \in \mathbb{C}^2 : z^3 = w^{3k+1}\}$ where $k \in \mathbb{N}$.

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- **Self-intersection, or singular point of *crossing* type**.
  - For example, let $\pi_1, \pi_2 \subset \mathbb{R}^4$ be two-dimensional planes such that $\pi_1 \cap \pi_2 = \{0\}$. Then $T = [\![\pi_1^+]\!] + [\![\pi_2]\!]$ is an area-minimizing current with boundary $\mathbb{R}$, and $0 \in \mathrm{Sing}_b(T)$.

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  - Alternatively, add to $\{(z,w) \in \mathbb{C}^2 : z^2 = w^3\}$ a half plane $[\![\pi^+]\!]$, where $\pi \cap \mathbb{C}^2 = \{0\}$.

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Remark

The proof is reduced to the case when T is collapsed at the boundary Γ.

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Conjecture

When Γ is a closed analytic curve, and T is an area-minimizing current with $\partial T = [\![\Gamma]\!]$, then $\mathrm{Sing}_b(T)$ is discrete.

Analytic boundary: motivations

Theorem (Gulliver-Lesley 1973, Gulliver 1977, White 1997)

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There are a C^∞ simple closed curve $\Gamma \subset \mathbb{R}^4$ and an area-minimizing current T spanning Γ, such that $\mathrm{Sing}_b(T)$ has an accumulation point.

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Remark

This is due to the *failure of unique continuation at the boundary*.

Analytic boundary: setup of the linearized model

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Definition

We say that a pair $f = (f^+, f^-)$ is in the space $W^{1,2}(\mathbb{D}, \mathcal{A}_Q^{\pm})$ with interface (γ, φ), if

1. $f^+ \in W^{1,2}(\mathbb{D}^+, \mathcal{A}_{Q+1})$ and $f^- \in W^{1,2}(\mathbb{D}^-, \mathcal{A}_Q)$;

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2. $f^+|_\gamma = f^-|_\gamma + [\![\varphi]\!]$.

Dirichlet energy-minimizers with analytic interface

Theorem (DeLellis-Z.)

Given an analytic interface (γ, φ)*, suppose* $f \in W^{1,2}(\mathbb{D}, \mathcal{A}_Q^{\pm})$ *minimizes the Dirichlet energy among all competitors with the prescribed interface. Then the singular set of* f *is discrete.*

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Remark

Exceptional case: non-homogeneous blow-down of half of the Enneper surface.

Thank you!