Model-theoretic consequences of $\mathrm{MIP}^* = \mathrm{RE}$

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Hot Topics: MIP* = RE and the Connes' Embedding Problem
SL Math
October 19, 2023

Overview

- From $\mathrm{MIP}^* = \mathrm{RE}$, we know that there is no algorithm such that, upon inputs the parameters for a nonlocal game $\mathfrak{G}$, enumerates a sequence of upper bounds to the quantum entangled value $\mathrm{val}^*(\mathfrak{G})$.
- In this talk, we show how this fact can be used to derive other undecidability results in operator algebras.
- These results will be based on *first-order languages* used for expressing properties about these algebras.

The language for tracial von Neumann algebras

- One defines **formulae** in the language of tracial von Neumann algebras by recursion on "complexity" of formulae:
  - **Atomic formulae:** $\tau(p(\vec{x}))$, where $p(\vec{x})$ is a $*$-polynomial. (Technically $\Re(\tau(p(\vec{x})))$ and $\Im(\tau(p(\vec{x})))$.)
  - Given formulae φ_1 and φ_2, $\frac{\varphi_1}{2}$ and $\varphi_1 \dot{-} \varphi_2$ are also formulae.
  - Given a formula φ and a variable x, $\sup_x \varphi$ and $\inf_x \varphi$ are formulae. **"quantifiers"**
- Technically, we have different kinds of variables for different operator norm balls.
- If $\varphi(\vec{x})$ is a formula, (M, τ) is a tracial von Neumann algebra, and $\vec{a} \in M$, then we can **interpret** the formula, obtaining $\varphi^M(\vec{a}) \in \mathbb{R}$.
- A **sentence** is a formula without free variables. A **theory** is a collection of sentences. Write $M \models T$ if $\sigma^M = 0$ for all $\sigma \in T$.
- A sentence is **universal** if it is of the form $\sup_{\vec{x}} \varphi(\vec{x})$ with $\varphi(\vec{x})$ quantifier-free.

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Universal theories

- Note that if σ is a universal sentence, then $\sigma^M = \sigma^{M^{\mathcal{U}}}$ for any ultrapower $M^{\mathcal{U}}$ of M. (Actually true for all sentences: Łos' theorem)
- In particular, if N embeds into $M^{\mathcal{U}}$, then $\sigma^N \leq \sigma^M$.
- Conversely: if $\sigma^N \leq \sigma^M$ for all universal sentences σ, then N embeds into an ultrapower of M.
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Definable sets

- Occasionally we will want to quantify over closed sets besides operator norm balls.
- This is only possible if the set X we want to quantify over is a **definable set**.
- This means that X is the zeroset of a formula φ such that, given any $\epsilon > 0$, there is $\delta > 0$ such that, if $\varphi(\vec{a}) < \delta$, then there is $\vec{b} \in X$ such that $d(\vec{a}, \vec{b}) \leq \epsilon$.
- If X is a definable set, then quantifications over X can be approximated by official formulae and this approxiomation is effective if the modulus $\epsilon \mapsto \delta$ is effective.

Lemma (Paulsen, Kim, and Schafhauser)

For each n, the set of PVMs $(e_1, \ldots, e_n)$ in $\mathcal{R}$ of length n form a definable subset of $\mathcal{R}^n$ with effective modulus.

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Other languages

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- The only thing that changes is what is considered an atomic formula:
  - The language of C^*-algebras: $\|p(\vec{x})\|$
  - The language of tracial C^*-algebras: $\|p(\vec{x})\|$ and $\tau(p(\vec{x}))$
  - The language of pairs of C^*-algebras: “two copies” of the language of C^*-algebras (two kinds of variables)

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The completeness theorem

- Gödel's classical completeness theorem relates the "semantic" notion of logical implication $\models$ and the "syntactic" notion of provability $\vdash$.
- Here is the continuous logic version of this:

Theorem (Pavelka-style completeness)

For any theory T and any sentence σ, we have

$$\sup\{\sigma^M : M \models T\} = \inf\{r \in \mathbb{Q}^{>0} : T \vdash \sigma \dot{-} r\}.$$

- Key point: if T is effectively enumerable, then so is the set of σ for which $T \vdash \sigma$.

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1 Background in logic

2 A Gödelian refutation of CEP

3 QWEP C^*-algebras

4 Tsirelson pairs of C^*-algebras

CEP and computability

Theorem (G. and Hart (2016))

If CEP holds, then there is an algorithm such that, upon input any universal sentence σ in the language of tracial von Neumann algebras, enumerates a sequence of upper bounds for $\sigma^{\mathcal{R}}$.

Proof.

- There is an effectively enumerable theory T_{II_1} in the language of tracial von Neumann algebras whose models are exactly the II_1 factors.
- By the completeness theorem,

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- The LHS$= \sigma^{\mathcal{R}}$ by CEP and the RHS is effectively enumerable.

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s-val$^*(\mathfrak{G})$ as a universal sentence

Theorem (Kim, Paulsen, Schafhauser)

$p \in C_{qa}^s(k,n)$ if and only if there are PVMs $e^1, \ldots, e^k$ of length n in $\mathcal{R}^{\mathcal{U}}$ such that $p(a,b|x,y) = \tau(e_a^x e_b^y)$.

Given a nonlocal game $\mathfrak{G}$, let $\psi_{\mathfrak{G}}(x_{v,i})$ denote the formula

$$\sum_{v,w} \mu(v,w) \sum_{i,j} D(v,w,i,j)\,\mathrm{tr}(x_{v,i}x_{w,j}).$$

Corollary

For any game $\mathfrak{G}$, we have

$$\text{s-val}^*(\mathfrak{G}) = \left(\sup_{x_{v,i} \in X_{n,k}} \psi_{\mathfrak{G}}(x_{v,i})\right)^{\mathcal{R}}.$$

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If CEP held, then letting $\sigma_{\mathfrak{G}}$ denote the "universal sentence" from the previous slide (really effective approximations), we could effectively enumerate upper bounds for $\text{s-val}^*(\mathfrak{G})$, contradicting $\text{MIP}^* = \text{RE}$. □

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There is a sequence $M_1, M_2, \ldots,$ of separable II_1 factors, none of which embed into an ultrapower of $\mathcal{R}$, and such that, for all $i < j$, M_i does not embed into an ultrapower of M_j.

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- Let M_1 be any counterexample to CEP.
- Let σ_1 be a universal sentence such that $\sigma_1^{\mathcal{R}} = 0$ but $r_1 := \sigma_1^{M_1} > 0$.
- Let $T_1 := T_{II_1} \cup \{\sigma_1 \dot{-} \frac{r_1}{2}\}$.
- Take $M_2 \models T_1$ such that M_2 does not embed into $\mathcal{R}^{\mathcal{U}}$.
- Since $\sigma_1^{M_2} \leq \frac{r}{2} < \sigma_1^{M_1}$, we have that M_1 does not embed into $M_2^{\mathcal{U}}$.
- Let $T_2 := T_1 \cup \{\sigma_2 \dot{-} \frac{r_2}{2}\}$ and take $M_3 \models T_2$ that does not embed into $\mathcal{R}^{\mathcal{U}}$...

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There is a sequence $M_1, M_2, \ldots,$ of separable II_1 factors, none of which embed into an ultrapower of $\mathcal{R}$, and such that, for all $i < j$, M_i does not embed into an ultrapower of M_j.

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The class of counterexamples to CEP is not closed under ultraproducts.

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- Suppose, towards a contradiction, that the class of counterexamples to CEP is closed under ultraproducts.
- The assumption implies that there is a universal sentence σ and $r > 0$ such that $\sigma^{\mathcal{R}} = 0$ and $\sigma^{M} \geq r$ for all counterexamples M to CEP.
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- There are type III versions of $\mathcal{R}$: the hyperfinite type III_1 factor $\mathcal{R}_\infty$ and for each $\lambda \in (0,1)$, the hyperfinite type III_λ factor $\mathcal{R}_\lambda$.
- To study them model theoretically, they need to be equipped with a (faithful, normal) state.
  - For $\mathcal{R}_\infty$, the choice of state is irrelevant. (Connes-Stormer transitivity)
  - $\mathcal{R}_\lambda$ has a distinguished **Powers state** φ_λ.
- By results of Ando, Haagerup, and Winslow, these algebras play the role of $\mathcal{R}$ in a type III version of CEP.

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An open problem

- We could use the previous ideas to prove some results about C^*-algebras.
- For example, the negative resolution of CEP is known to imply a negative resolution to the MF problem: does every stably finite C^*-algebra embed into $\mathcal{Q}^{\mathcal{U}}$, where $\mathcal{Q}$ is the **universal UHF algebra**? We can give Gödelian refutations to the MF problem and other such problems...
- However, all of these applications use that these algebras have traces and we can “interpret” the WOT closure in the GNS to apply our tracial von Neumann algebra results.
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Introducing QWEP

- We say that A has the **weak expectation property (WEP)** if, for any $B \supseteq A$ and C, the natural map $A \otimes_{\max} C \to B \otimes_{\max} C$ is an isometric inclusion.
- A has the **QWEP property** if A is a quotient of a C^*-algebra with the WEP.
- Kirchberg proved that all C^*-algebras have the QWEP if and only if CEP holds.
- A key ingredient: a tracial von Neumann algebra has QWEP if and only if it satisfies CEP.

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QWEP is not effectively axiomatizable

Theorem (Arulseelan, G., and Hart)

There is no effectively enumerable theory T in the language of C^*-algebras with the following two properties:

1. *All models of T have QWEP.*
2. *There is an infinite-dimensional, monotracial model A of T whose unique trace is faithful.*

In particular, there is no effective theory T in the language of C^*-algebras that axiomatizes the QWEP C^*-algebras.

Proof of the theorem

- Suppose, TAC, that such T existed. Take an infinite-dimensional, monotracial model A of T whose unique trace τ_A is faithful.
- Work now in the language of tracial C^*-algebras and consider the theory T' consisting of the axioms for tracial C^*-algebras together with T. Note that T' is effective and $(A, \tau_A) \models T'$.
- Note that, for any universal sentence σ in the language of tracial von Neumann algebras, we have

$$\sup\{\sigma^{(B,\tau_B)} \ : \ (B, \tau_B) \models T'\} = \sigma^{(\mathcal{R},\tau_{\mathcal{R}})}.$$

  - $\geq$: If $(M, \tau_M) = \text{GNS}(A, \tau_A)$, then $A \subseteq M$ and (M, τ_M) is a II_1 factor, so $\sigma^{(A,\tau_A)} = \sigma^{(M,\tau_M)} \geq \sigma^{(\mathcal{R},\tau_{\mathcal{R}})}$.
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Motivating the definition of the Tsirelson property

Tsirelson's Problem

Does $C_{qa}(k,n) = C_{qc}(k,n)$?

Theorem

$p \in C_{qa}(k,n)$ (resp. $p \in C_{qc}(k,n)$) if and only if there are POVMs A^x and B^y in $\mathrm{C}^*(\mathbb{F}(k,n))$ and a state ϕ on $\mathrm{C}^*(\mathbb{F}(k,n)) \otimes_{\min} \mathrm{C}^*(\mathbb{F}(k,n))$ (resp. on $\mathrm{C}^*(\mathbb{F}(k,n)) \otimes_{\max} \mathrm{C}^*(\mathbb{F}(k,n))$) such that

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Definition

Let $C_{\min}(C, D, k, n)$ (respectively $C_{\max}(C, D, k, n)$) denote the *closure* of the set of correlations of the form $\phi(A^x_a \otimes B^y_b)$, where $A^1, \ldots, A^k$ are POVMs of length n from C, $B^1, \ldots, B^k$ are POVMs of length n from D, and ϕ is a state on $C \otimes_{\min} D$ (respectively a state on $C \otimes_{\max} D$).

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We say that (C, D) is a **(strong) Tsirelson pair** if $C_{\min}(C, D, k, n) = C_{\max}(C, D, k, n)$ $(=C_{qa}(k, n))$ for all (k, n).

About Tsirelson pairs

- Tsirelson's problem asks if $(C^*(\mathbb{F}_\infty), C^*(\mathbb{F}_\infty))$ is a Tsirelson pair. We now know that it is not.
- If (C, D) is a **nuclear pair**, that is, if $C \otimes_{\min} D \cong C \otimes_{\max} D$, then (C, D) is a Tsirelson pair.
- Exactly one of the following happens:
  - (C, D) is not a Tsirelson pair.
  - One of C or D is **subhomogeneous** (whence (C, D) is a nuclear pair), but (C, D) is not a strong Tsirelson pair.
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C^*-algebras with the Tsirelson property

Definition

C has the **Tsirelson property (TP)** if (C, D) is a Tsirelson pair for any C^*-algebra D.

- C has the TP if and only if $(C, C^*(\mathbb{F}_\infty))$ is a Tsirelson pair.
- The class of C^*-algebras with TP is closed under direct limits, quotients, **relatively weakly injective** subalgebras, and ultraproducts. In particular, it is an *axiomatizable* class.
- QWEP implies TP. (Proof: ETS WEP implies TP; but C has WEP if and only if $(C, C^*(\mathbb{F}_\infty))$ is a nuclear pair.)

Question

Does TP imply QWEP?

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Definition

C has the **strong Tsirelson property (STP)** if and only if it has the TP and is not subhomogeneous.

- C has the STP if and only if (C, D) is a strong Tsirelson pair for every non-subhomogeneous D.
- The STP is an axiomatizable property.

Question

Are there explicit axioms for the class of C^*-algebras with the (S)TP?

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Another undecidability result

Theorem (G. and Hart)

There is no effective theory T in the language of pairs of C^*-algebras such that all models of T are Tsirelson pairs and at least one model of T is a strong Tsirelson pair.

Corollary

There is no effective theory T in the language of C^*-algebras such that all models have the TP and at least one model has the STP.

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There is no effective theory T in the language of C^*-algebras such that all models have the QWEP and at least one model is not subhomogeneous.

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There is no effective theory T in the language of C^*-algebras such that all models have the QWEP and at least one model is not subhomogeneous.

Proof of the previous theorem

- Suppose such T exists.
- Let T' be the effective extension of T whose models are of the form (C, D, P), where $P(c, d) = \phi(c \otimes d)$ for some state ϕ on $C \otimes_{\max} D$.
  - States on $C \otimes_{\max} D$ "are" just extensions of unital linear functionals on $C \odot D$ that are positive on $C \odot D$.
- Given a nonlocal game $\mathfrak{G}$, have the universal sentence $\sigma_{\mathfrak{G}}$ in this extended language given by

$$\sup_A \sup_B \sum_{(x,y)\in[k]} \pi(x,y) \sum_{(a,b)\in[n]} D(x,y,a,b) P(A_a^x, B_b^y).$$

- The quantifications over POVMs here is legitimate (and effective) since they can be shown to form a definable set.

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  - $\leq$ uses that all models of T are Tsirelson pairs.
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- This contradicts MIP* =RE.

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